UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2009/2010

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let S be a set of real numbers. Explain what is meant by the following.

    i. A real number α is the maximum of S. [3 marks]

    ii. A real number β is the minimum of S. [3 marks]

   (b) Find, if they exist, the maximum and minimum of the set:
   $S := \{x \in \mathbb{R} : x + |x - 1| = 2 + |x|\}$ [4 marks]

   (c) Determine whether the following statements are true or false. Justify your answers.

    i. The sum of a rational number and an irrational number is always irrational. [3 marks]

    ii. There exists a set of real numbers with a maximum but no supremum. [3 marks]

   (d) Let A be a non-empty subset of $\mathbb{R}$ and let $f, g : A \to \mathbb{R}$ be functions, each with a bounded range in $\mathbb{R}$. Show that

$$\inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\} \leq \inf\{f(x) + g(x) : x \in A\}$$

[4 marks]

QUESTION 2

2. (a) Let (x_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Explain precisely what is meant by the statement

$$x_n \to l \text{ as } n \to \infty$$

[3 marks]

   Use this definition to show that

    i.
$$\frac{2n-1}{3n+8} \to \frac{2}{3} \text{ as } n \to \infty$$

[4 marks]

    ii. Starting from the theorem that a sequence of real numbers that is convergent is also bounded, show that if

$$x_n \to l \text{ as } n \to \infty \text{ and } y_n \to m \text{ as } n \to \infty$$

    then $x_n y_n \to lm$ as $n \to \infty$. [5 marks]

(b) Consider the sequence (x_n) defined by

$$x_1 = 2, \ \ 4x_{n+1} = x_n^2 + 3, \text{ for } n \geq 1$$

i. Show that $1 < x_n < 3$ for all $n \geq 1$. [3 marks]

ii. Prove that (x_n) is a decreasing sequence. [3 marks]

iii. Deduce that (x_n) is convergent and find its limit. State any theorem used. [2 marks]

QUESTION 3

3. (a) i. Explain what it means to say that a function $f : [a, b] \to \mathbb{R}$ is continuous at a point $c \in (a, b)$. [2 marks]

ii. Give an example of a function $f : [-1, 1] \to \mathbb{R}$ which is not continuous and yet the function $|f| : [-1, 1] \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$ is continuous. [2 marks]

iii. Use the definition of $\lim_{x \to a} f(x)$ to show that

$$\lim_{x \to 0} \frac{x^2}{|x|} = 0, \ \ (x \neq 0)$$

[5 marks]

(b) State the Intermediate Value theorem and use it to show that the equation

$$x^2 = \cos x$$

has a solution in the interval $(0, \frac{\pi}{2})$. [5 marks]

(c) Determine whether the following statements are true or false. Justify your answers.

i. All functions $f : [-1, 1] \to [-1, 1]$ satisfy $f(x) = x$ at any $x \in [-1, 1]$. [3 marks]

ii. There is a continuous function $f : [-1, 0) \to \mathbb{R}$ which does not attain a maximum value on $[-1, 0)$. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \longrightarrow \mathbb{R}$ be a function.

   i. Explain what is meant by saying that f is differentiable at $c \in (a, b)$. [2 marks]

   ii. Show that if f is differentiable at $c \in (a, b)$ then f is continuous at point c. [4 marks]

   iii. Is the converse of part 4(a)ii above true? Justify your answer. [2 marks]

(b) i. State the Mean Value Theorem. [2 marks]

   ii. Let $f : [a, b] \longrightarrow [a, b]$ be differentiable on $[a, b]$. Also, let

$$|f'(x)| < 1, \ \forall x \in [a, b]$$

   Then, show that

$$|f(x) - f(y)| < |x - y|, \forall x, y \in [a, b]$$

[4 marks]

(c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x - 1, & \text{if } x \leq 1 \\ \ln x, & \text{if } x > 1 \end{cases}$$

   i. Show that f is differentiable at $x = 1$. [4 marks]

   ii. Is f continuous at $x = 1$? Justify your answer. [2 marks]

QUESTION 5

5. (a) Use the Mean Value theorem to show that

   i. $\frac{b-a}{2\sqrt{b}} < \sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}$ for $0 < a < b$. [4 marks]

   ii. $1 - \frac{a}{b} < \ln \frac{b}{a} < \frac{b}{a} - 1$ for $0 < a < b$. [5 marks]

   iii. $1 + x < e^x < \frac{1}{1-x}$ for $0 < x < 1$. [6 marks]

   (b) Let a sequence (x_n) of real numbers be defined by

$$x_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \text{ for } n \geq 1$$

   Use part 5a above or otherwise to show that $x_{n+1} < x_n$ for $n \geq 1$. [5 marks]

QUESTION 6

6. (a) Let $\sum a_n$ be a series in the set $\mathbb{R}$ of real numbers. Then, explain what is meant by the following statements.

   i. The k-th partial sum. [2 marks]

   ii. $\sum a_n$ converges. [2 marks]

   iii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) Prove that if $\sum a_n$ converges, then $\lim(a_n) = 0$. [3 marks]

   (c) Use part 6b above to show that

$$\sum \frac{n}{n+1}$$

   diverges. [2 marks]

   (d) Starting from the Cauchy convergence criterion prove that:
   If $\sum a_n$ is absolutely convergent, then $\sum a_n$ converges. [6 marks]

   (e) For the series

$$1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

   determine whether it converges or diverges. State any theorems used. [4 marks]

QUESTION 7

7. (a) Explain in detail what it means for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$. [10 marks]

(b) Use part 7a above to show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} +1, & \text{if } x \text{ is rational} \\ -1, & \text{otherwise} \end{cases}$$

is **not** Riemann integrable. [6 marks]

(c) For each of the following statements prove if true otherwise give a counterexample.

i. Every bounded function $f : [0, 1] \to \mathbb{R}$ is Riemann integrable. [2 marks]

ii. There is a function $f : [0, 1] \to \mathbb{R}$ such that the function $|f| : [0, 1] \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$ is Riemann integrable, but f is not Riemann integrable. [3 marks]