UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2009/2010

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let S be a set of real numbers. Explain what is meant by the following.

   i. A real number α is the supremum for S. [3 marks]

   ii. A real number β is the infimum for S. [3 marks]

   (b) Find, if they exist, the supremum and infimum of the set:
   $S := \{x \in \mathbb{R} : |x - 1| = |x|\}$ [4 marks]

   (c) Decide whether the following statements are true or false. Justify your answers.

   i. There exists a set of real numbers with a minimum but no infimum. [3 marks]

   ii. The sum of a rational number and an irrational number is always irrational. [3 marks]

   (d) Let A be a non-empty subset of $\mathbb{R}$ and let $f, g : A \to \mathbb{R}$ be functions, each with a bounded range in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}$$

[4 marks]

QUESTION 2

2. (a) Let (x_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Explain precisely what is meant by the statement

$$x_n \to l \text{ as } n \to \infty$$

[3 marks]

Use this definition to show that

i.

$$\frac{n}{n+1} \to 1 \text{ as } n \to \infty$$

[4 marks]

ii. Show that if

$$x_n \to l \text{ as } n \to \infty \text{ and } y_n \to m \text{ as } n \to \infty$$

then $x_n - y_n \to l - m$ as $n \to \infty$. [5 marks]

(b) Consider the sequence (x_n) defined by

$$x_1 = 10, \quad x_{n+1} = \frac{x_n^2 - 8}{2x_n - 6}, \text{ for } n \geq 1$$

i. Show that $x_n \geq 4$ for all $n \geq 1$. [3 marks]

ii. Prove that (x_n) is a decreasing sequence. [3 marks]

iii. Deduce that (x_n) is convergent and find its limit. State any theorem used. [2 marks]

QUESTION 3

3. (a) i. Explain what it means to say that a function $f : [a, b] \to \mathbb{R}$ is continuous at a point $c \in (a, b)$. [2 marks]

ii. Give an example of a function $f : [-1, 1] \to \mathbb{R}$ which is not continuous at $x = 0$. [2 marks]

iii. Use the definition of $\lim_{x \to a} f(x)$ to show that

$$\lim_{x \to 0}(|x| + x) = 0$$

[5 marks]

(b) Use the Intermediate Value theorem to show that the equation

$$e^x + x = 2$$

has a solution in the interval $(0, 1)$. [5 marks]

(c) Decide whether the following statements are true or false. Justify your answers.

i. All bounded functions $f : [0, 1] \to \mathbb{R}$ are continuous. [3 marks]

ii. There is a continuous function $f : (0, 1) \to \mathbb{R}$ which does not attain a minimum value on $(0, 1)$. [3 marks]

QUESTION 4

4. (a) Let $f : (a,b) \to \mathbb{R}$ be a function.

   i. What does it mean to say that f is differentiable at $c \in (a,b)$. [2 marks]

   ii. Show that if f is differentiable at $c \in (a,b)$ then f is continuous at point c. [4 marks]

   iii. Give an example of a function $f : (-1,1) \to \mathbb{R}$ that is continuous at $x = 0$ but not differentiable there. [2 marks]

(b) i. State the Mean Value Theorem. [2 marks]

   ii. Let $f : [a,b] \to [a,b]$ be differentiable on $[a,b]$. Also, let

$$|f'(x)| \leq K, \ \forall x \in [a,b]$$

and for some $K \in \mathbb{R}$. Then, show that

$$|f(x) - f(y)| \leq K|x-y|, \forall x, y \in [a,b]$$

[4 marks]

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 2x, & \text{if } x \leq 0 \\ (x+1)^2, & \text{if } x > 0 \end{cases}$$

   i. Show that f is differentiable at $x = 0$. [4 marks]

   ii. Is f continuous at $x = 0$? Justify your answer. [2 marks]

QUESTION 5

5. (a) Let $f : [a, b] \to \mathbb{R}$ be a function that is differentiable on (a, b). If $f'(x)$ is non-negative on (a, b), then use the Mean Value Theorem or otherwise to show that f is increasing on (a, b). [6 marks]

   (b) Use the Mean Value theorem to show that

   i. $\frac{1}{4} < 4 - \sqrt{14} < \frac{1}{3}$. [4 marks]

   ii. $\frac{1}{2} < \ln 2 < 1$. [4 marks]

   iii. $1 + x < e^x < \frac{1}{1-x}$ for $0 < x < 1$. [6 marks]

QUESTION 6

6. (a) Let $\sum a_n$ be a series in the set $\mathbb{R}$ of real numbers. Then, explain what is meant by the following statements.

   i. The k-th partial sum. [2 marks]

   ii. $\sum a_n$ converges. [2 marks]

   iii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) Prove that if $\sum a_n$ converges, then $\lim(a_n) = 0$. [3 marks]

   (c) Use part 6b above to show that

$$\sum \left(1 + \frac{1}{n}\right)$$

   diverges. [2 marks]

   (d) Starting from the Cauchy convergence criterion prove that:
   If $\sum a_n$ is absolutely convergent, then $\sum a_n$ converges. [6 marks]

   (e) Determine whether the series

$$\sum (-1)^{n+1} \frac{1}{n}$$

   it converges or diverges. State any theorems used. [4 marks]

QUESTION 7

7. (a) Explain in detail what is meant by saying that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. [10 marks]

   (b) Use part 7a above to determine whether or not the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & \text{if } x \text{ is rational} \\ 2, & \text{otherwise} \end{cases}$$

   is Riemann integrable. [6 marks]

   (c) Decide whether the following statements are true or false. Justify your answers.

      i. There is a bounded function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable. [2 marks]

      ii. There is an Riemann integrable function $f : [0, 1] \to \mathbb{R}$ and a function $g : [0, 1] \to \mathbb{R}$ which is not Riemann integrable such that the product function $fg : [0, 1] \to \mathbb{R}$ is Riemann integrable. [3 marks]