UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2009/2010

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER : NUMERICAL ANALYSIS II

COURSE NUMBER : M 411

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :

1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS.
3. NON PROGRAMMABLE CALCULATORS MAY BE USED.

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Approximate e^x on $[-1, 1]$ using a linear least squares polynomial.

[10 marks]

(b) For the data

i	0	1	2	3
x_i	1	2	5	7
y_i	4	6	7	8

if the problem is to find a quadratic polynomial that best fits the data in the least squares sense, then deduce the normal equations in matrix form

$$A\mathbf{a} = \mathbf{b} \qquad (1)$$

Explicitly dentify the vectors $\mathbf{a}$ and $\mathbf{b}$, and the square matrix A.
DO NOT solve the linear system (1). [10 marks]

QUESTION 2

2. (a) Show that the Chebyshev polynomials $\{T_0(x), T_1(x), T_2(x), \dots\}$ of the first kind are orthogonal on the open interval $(-1, 1)$ with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$. [10 marks]

(b) Use the Gram-Schmidt process to find the degree 0 to 1 orthogonal polynomials ($\phi_0(x)$ and $\phi_1(x)$ respectively) in the closed interval $[1, 2]$ with respect to the weight function $w(x) = x$.
If $p(x) := a_0\phi_0(x) + a_1\phi_1(x)$ is a least squares approximation to $f(x) = x^2 + 2x - 3$, then determine the constant a_0. [10 marks]

QUESTION 3

3. Use a single step of the modified Euler method to solve the Initial Value problem:

$$x'' - 2x' + x = te^t - t,\ 0 \le x \le 1,\ x(0) = 0, x'(0) = 1,$$

for $x(0.1)$ and $x'(0.1)$. [20 marks]

QUESTION 4

4. (a) Solve the initial value problem

$$y'(x) = -y + x\sqrt{y},\ 2 \leq x \leq 4,\ y(2) = 2$$

for $y(2.1)$ using one step of each of the following;

i. Taylor series method of order 2. [3 marks]

ii. The single step Adams-Bashforth method. [3 marks]

iii. The Runge-Kutta method of order 2. [3 marks]

(b) Given the linear multi-step method

$$y_{n+1} = -\frac{3}{2}y_n + 3y_{n-1} - \frac{1}{2}y_{n-2} + 3hf_n$$

for solving the initial value problem

$$y'(x) = f(x, y),\ a \leq x \leq b,\ y(a) = \alpha$$

determine whether or not it is convergent. [11 marks]

QUESTION 5

5. (a) Solve the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= x + y,\ 0 < x < 1,\ 0 < y < 1, \\ u(0, y) &= 1,\ u(1, y) = 1 - y,\ 0 \leq y \leq 1, \\ u(x, 0) &= 1,\ u(x, 1) = 1 - x,\ 0 \leq y \leq 1, \end{aligned}$$

using the Finite Difference method on a rectangular grid with a stepsizes $h = \dfrac{1}{3}$ and $k = \dfrac{1}{2}$ in the x and y directions respectively. [12 marks]

(b) Let α and β be constants. Suppose that the finite difference scheme

$$U_j^{n+1} = \alpha U_j^n + \beta U_{j-1}^n$$

is used to approximate the solution of the advection equation

$$u_t + au_x = 0,$$

where $a > 0$ is a constant and U_j^n approximates $u(x_j, t_n)$ in the usual notation. Then, determine both α and β so that the scheme agrees with the Taylor series expansion of $u(x_n, t_{n+1})$ to as high an order as possible. [8 marks]

QUESTION 6

6. Consider the differential problem;

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \\ u_x(0,t) =& u(0,t),\ u(1,t) = 1, t > 0, \\ u(x,0) =& x(1-x),\ 0 \le x \le 1. \end{aligned} \tag{2}$$

Suppose that the parabolic diffusion equation (2) is approximated by replacing u_t with a forward difference, and that u_{xx} is replaced by a central difference. Also, suppose that the derivative u_x in the boundary condition at $x = 0$ is replaced by a forward difference approximation. Then.

(a) Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{u}^{(n+1)} = B\mathbf{u}^{(n)} + \mathbf{v}, \text{ where } n = 1, 2, \ldots$$

Identify the square matrix B, and the vectors $\mathbf{u}^{(n)}$ and $\mathbf{v}$. [12 marks]

(b) Compute the leading terms of the truncation error for this numerical scheme. [8 marks]

QUESTION 7

7. Consider the differential problem;

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \\ u_x(0,t) =& u(1,t) = 0, t > 0, \\ u(x,0) =& \sin \pi x,\ 0 \le x \le 1. \end{aligned}$$

(a) Deduce the fully implicit numerical scheme resulting from using a backward difference approximation for the derivative u_t, and a central difference approximation for the derivative u_{xx}. [10 marks]

(b) Prove that the scheme is unconditionally stable. [10 marks]