UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2009/2010

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER : NUMERICAL ANALYSIS II

COURSE NUMBER : M 411

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :
1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS.
3. NON PROGRAMMABLE CALCULATORS MAY BE USED.

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Approximate x^2e^x on $[0,1]$ using a linear least squares polynomial. [10 marks]

   (b) Find a linear polynomial that best fits the data

i	0	1	2	3
x_i	1	2	5	7
y_i	4	6	7	8

   in the least squares sense. [10 marks]

QUESTION 2

2. (a) For the Chebyshev polynomials $\{T_0(x), T_1(x), T_2(x), \dots\}$ prove the following properties given that i and j are positive integers.

      i. $T_i(T_j(x)) = T_{ij}(x)$. [4 marks]

      ii. If $j > i$, then $T_i(x)T_j(x) = \frac{1}{2}[T_{i+j}(x) + T_{j-i}(x)]$. [6 marks]

   (b) Prove that the trigonometric functions $\{\cos nx : n = 0, 1, 2, \dots\}$ are orthogonal on the closed interval $[0, 2\pi]$ with respect to the weight function $w(x) = 1$. [10 marks]

QUESTION 3

3. Use a single step of the modified Euler method to solve the Initial Value problem:

$$x'' - 3x' + 2x = 6e^{-t},\ 0 \leq x \leq 1,\ x(0) = x'(0) = 2,$$

for both $x(0.1)$ and $x'(0.1)$. [20 marks]

QUESTION 4

4. (a) Consider initial value problem

$$y' = 1 - y,\ 0 \le t \le 1,\ y(0) = 0$$

i. Use one step of the Euler method to compute $y(0.1)$. [3 marks]

ii. Compute $y(0.2)$ using the linear multi-step method

$$y_{n+1} = y_n + \frac{h}{2}[f_{n-1} + f_n]$$

using both $y(0) = 0$ and the value of $y(0.1)$ from 4(a)i. [3 marks]

iii. Use another step of method in 4(a)ii to compute $y(0.3)$. [3 marks]

(b) Discuss the convergence of the linear multi-step method

$$y_{n+2} = -\ 3y_n + 4y_{n+1} - 2hf_n\,.$$

[11 marks]

QUESTION 5

5. (a) For the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= xy,\ 0 < x < 1,\ 0 < y < 1, \\ u(0,y) &= 0,\ u(1,y) = y,\ 0 \le y \le 1, \\ u(x,0) &= 0,\ u(x,1) = x,\ 0 \le y \le 1, \end{aligned}$$

replace the derivatives u_{xx} and u_{yy} with central difference approximations, and use a uniform grid with step size $h = \frac{1}{2}$ to compute the approximate value of the solution u at $(x, y) = (1, 1)$. [10 marks]

(b) Consider the wave equation

$$u_{tt} = u_{xx},\ 0 < x < 1,\ t > 0,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0,\ t > 0,$$

and initial conditions

$$u(x,0) = x(1-x),\ u_t(x,0) = 0,\ 0 \le x \le 1.$$

Write down the corresponding finite difference problem based on central difference approximation of the derivatives. [10 marks]

QUESTION 6

6. Consider the differential problem;

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \qquad (1)\\ u(0,t) &= 0,\ u_x(1,t) = u(1,t) - 1, t > 0,\\ u(x,0) &= x(1-x),\ 0 \le x \le 1. \end{aligned}$$

Suppose that the parabolic diffusion equation (1) is approximated by replacing u_t with a backward difference, and that u_{xx} is replaced by a central difference. Also, suppose that the derivative u_x in the boundary condition at $x = 1$ is replaced by a backward difference approximation. Then,

(a) Show that the resulting finite difference equations may be written in matrix form as

$$A\mathbf{u}^{(n)} = \mathbf{u}^{(n-1)} + \mathbf{v},\ \text{ where } n = 1, 2, \ldots$$

Identify the square matrix A, and the vectors $\mathbf{u}^{(n)}$ and $\mathbf{v}$. [12 marks]

(b) Compute the leading terms of the truncation error for this numerical scheme. [8 marks]

QUESTION 7

7. Consider the differential problem;

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t),\ 0 < x < 1,\ t > 0,\\ u(0,t) &= u(1,t) = 0, t > 0,\\ u(x,0) &= \sin \pi x,\ 0 \le x \le 1. \end{aligned}$$

(a) Deduce the fully explicit numerical scheme resulting from using a forward difference approximation for the derivative u_t, and a central difference approximation for the derivative u_{xx}. [10 marks]

(b) Prove that the scheme is stable provided

$$\nu := \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2},$$

where Δt and Δx are the step sizes in the t and x directions respectively. [10 marks]