University of Swaziland

Final Examination, December 2009

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations
Course Number : M415
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.
5. A Table of Laplace Transforms is provided at the end of the question paper.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Consider the expression

$$u = x^2 f(y^2 u) \qquad (1)$$

where $u = u(x, y)$ and f is an arbitrary function.
Find the partial differential equation for which (1) is a general solution. [10 marks]

(b) Find the integral of

$$(y + u)u_x + (x + u)u_y = x + y$$

through the curve $u = 0, \quad y = 2x$. [10 marks]

Question 2

Consider the equation

$$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + 2x u_x = 0. \qquad (2)$$

a) Classify (2) as hyperbolic, parabolic or elliptic. [2 marks]

b) Reduce (2) into its canonical form. [13 marks]

c) Hence find the general solution of (2). [5 marks]

Question 3

Consider the function $f(x) = x(\pi - x), \quad 0 \leqslant x \leqslant \pi$.

a) Show that $f(x)$ can be represented by the sine series

$$\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}.$$

b) Using 3. a), or otherwise, deduce the identity

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\cdots=\frac{\pi^3}{32}.$$

c) Use Parseval's Identity to find the value of the sum

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}.$$

Question 4

Solve the boundary-value problem [20 marks]

$$\begin{aligned} u_t-2u_{xx} &= e^{-2t}\sin\tfrac{3}{2}x, && 0<x<\pi,\ t>0,\\ u(x,0) &= 4\sin\tfrac{3}{2}x, && 0\le x\le\pi,\\ u(0,t) &= u_x(\pi,t)=0, && t\ge 0. \end{aligned}$$

Question 5

Find the solution of the steady-state problem [20 marks]

$$\begin{aligned} u_{xx}+u_{yy} &= 0, && 0<x<\pi,\ 0<y<\pi,\\ u(0,y) &= 8\sin 2y, && 0\le y\le\pi,\\ u(\pi,y) &= 0, && 0\le y\le\pi,\\ u(x,0) &= u(x,\pi)=0, && 0\le x\le\pi. \end{aligned}$$

Question 6

The initial temperature distribution of a thin circular disk is T_0. If the disk is then allowed to cool down with the circula edge kept at temperature $u = 0$, the subsequent temperature distribution is governed by the system

$$\begin{aligned} u_t &= u_{rr} + \tfrac{1}{r}u_r, && 0 < r < 1,\ t > 0 \\ u(r,0) &= T_0, && 0 \leqslant r \leqslant 1, \\ u(1,t) &= 0, && t > 0. \end{aligned}$$

Solve for $u(r,t)$. [20 marks]

Question 7

Consider the problem

$$\begin{aligned} u_t + xu_x &= x, && x \geqslant 0,\ t > 0 \\ u(x,0) = u(0,t) &= 0. \end{aligned}$$

Solve for $u(x,t)$ using

a) the method of Laplace transforms [10 marks]

b) any other method. [10 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(at)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(at)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$