University of Swaziland

Supplementary Examination – July 2010

BSc IV, BASS IV, BEd IV

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Consider the expression

$$u = xy + f\left(x^2 - y^2\right) \qquad (1)$$

where $u = u(x, y)$ and f is an arbitrary function. Find the partial differential equation for which (1) is a general solution. [10 marks]

(b) Find the integral of

$$(x - y)u_x + yu_y = xu$$

through the curve $u = 1, \quad y = x$. [10 marks]

Question 2

Consider the equation

$$8u_{xx} + 2u_{xy} - 15u_{yy} = 484. \qquad (2)$$

a) Classify (2) as hyperbolic, parabolic or elliptic. [2 marks]

b) Reduce (2) into its canonical form. [13 marks]

c) Hence find the general solution of (2). [5 marks]

Question 3

Consider the Cauchy problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty,\ t \geq 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x), \quad -\infty < x < \infty. \end{aligned}$$

Derive the d'Alembert's solution

$$u(x,t) = \tfrac{1}{2}\Big\{f(x+ct) + g(x-ct)\Big\} + \tfrac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\mathrm{d}\alpha.$$

[20 marks]

Question 4

Solve the boundary-value problem [20 marks]

$$\begin{aligned} u_t - 2u_{xx} &= 0, \qquad 0 < x < \pi,\ t > 0, \\ u(x,0) &= 4\cos\tfrac{3}{2}x, \qquad 0 \le x \le \pi, \\ u_x(0,t) &= u(\pi,t) = 0, \qquad t \ge 0. \end{aligned}$$

Question 5

Find the solution of the steady-state problem [20 marks]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < \pi,\ 0 < y < \pi, \\ u(x,0) &= x(\pi - x), \qquad 0 \le x \le \pi, \\ u(x,\pi) &= 0, \qquad 0 \le x \le \pi, \\ u(0,y) &= u(\pi,y) = 0,\ 0 \le y \le \pi. \end{aligned}$$

Question 6

Solve the Dirichlet problem inside the circle

$$\begin{aligned} u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} &= 0,\ \ 0 < r < 1,\ \ -\pi < \theta < \pi, \\ u(1,\theta) &= 1 - \cos 2\theta,\ \ -\pi \le \theta \le \pi. \end{aligned}$$

[20 marks]

Question 7

The 2-D Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

a. Show that under the the transformation $x = \rho\cos\varphi,\ y = \sin\varphi$, the Laplacian becomes

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}.$$

b. Evaluate the Jacobian of the transformation, and hence comment on whether the transformation breaks down anywhere.

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(at)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(at)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$