University of Swaziland

Final Examination, 2009/10

BSc IV, Bass IV, BEd IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Give the formal definition of a Euclidean ring R. [4]

(b) Let R be a Euclidean ring.

i. Prove that any two elements a and b in R have a greatest common divisor d. [5]

ii. Show that there exists $m, n \in R$ such that $d = am + bn$. [5]

(c) Prove that every finite integral domain is a field. [6]

Question 2

(a) Give an example of a ring satisfying the given conditions (do not prove anything).

i) A ring without a unity. [2]

ii) A finite integral domain. [2]

iii) A ring that is not a division ring. [2]

iv) A ring with 0 divisors. [2]

(b) In a ring $\mathbb{Z}_n$, show that

i. zero divisors are those elements that are **not** co-prime. [5]

ii. elements that are co-prime cannot be zero divisors. [5]

(c) Describe all units in the ring

$$\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}.$$ [2]

Question 3

(a) Let $\varphi_\alpha : \mathbb{Z}[x] \to \mathbb{Z}_7$. Evaluate each of the following

i. $\varphi_5\left[(x^3+2)(4x^2+3)(x^7+3x^2+1)\right]$ [5]

ii. $\varphi_4\left[3x^{106}+5x^{99}+2x^{53}\right]$ [5]

(b) Show that the rings $\mathbb{Z}$ and $2\mathbb{Z}$ are not isomorphic. [4]

(c) Show that for a field F, the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{for } a, b \in F$$

is a right ideal but not a left ideal of the ring $R = M_2(F)$. [6]

Question 4

Which of the following are integral domains and which are fields? Justify your answer.

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$ [5]

(b) $\{a+ib : a, b \in \mathbb{Q}\}$ [5]

(c) $\mathbb{Z} \times \mathbb{R}$ [5]

(d) $\mathbb{R}[x]$ [5]

Question 5

(a) Determine which of the following polynomials in $\mathbb{Z}[x]$ satisfy an Eisenstein criterion for irreducibility over $\mathbb{Q}$

i. $4x^{10} - 9x^3 + 24x - 18$ [4]

ii. $2x^{10} - 25x^3 + 10x^2 - 30$ [4]

b. Express $f(x) = x^3 + 2x + 3$ over $\mathbb{Z}_5[x]$ as a product of irreducible polynomial over $\mathbb{Z}_5[x]$. [6]

c. Prove that, if D is an integral domain, then $D[x]$ is also an integral domain. [6]

Question 6

(a) Let α be a zero of x^2+x+1 in the extention field of $\mathbb{Z}_2$. Give the addition and multiplication tables for the four elements of $\mathbb{Z}_2(\alpha)$. [6]

(b) Show that the polynomial $f(x) = x^p + a$ in $\mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$. [5]

(c) For each of the given algebraic numbers $\alpha \in \mathbb{C}$, find $\ln(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$

i. $\sqrt{3 - \sqrt{6}}$ [3]

ii. $\sqrt{\frac{1}{3} + \sqrt{7}}$ [3]

iii. $\sqrt{2} + i$ [3]

Question 7

(a) Find all the monic irreducible polynomials of degree 2 over $\mathbb{Z}_3$. [9]

(b) Prove that every field is an integral domain. [7]

(c) Factor the polynomial $4x^2 - 4x + 8$ as a product of irreducibles in $\mathbb{Z}_{11}[x]$. [4]