University of Swaziland

Supplementary Examination, 2009/10

BSc IV, Bass IV, BEd IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Find the greatest common divisor of the polynomials $f(x) = x^4+4x^3+7x^2+6x+2$ and $g(x) = x^3+4x^2+7x+4$ over $\mathbb{Q}$ and express it as a linear combination of $f(x)$ and $g(x)$. [8]

(b) Prove that if R is a ring with unity and N is an ideal of R containing a unit, then $N = R$. [6]

(c) Describe all units in each of the following rings.

i. $\mathbb{Z}_7$

ii. $Z \times \mathbb{Q} \times \mathbb{Z}_3$

[6]

Question 2

(a) State whether or not the given function v is a Euclidean valuation for the given integral domain.

i) $v(n) = n^2$ for non-zero $n \in \mathbb{Z}$,

ii) $v(a) = 50$ for non-zero values of $a \in \mathbb{Q}$.

[8]

(b) State Kronecker's theorem. [Do not prove]. [4]

(c) Given that every element β of $E = F(\alpha)$ can be uniquely expressedd in the form $\beta = b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{n-1}\alpha^{n-1}$ where each $b_i \in F$, α algebraic over the field F and $\deg(\alpha, F) \geqslant 1$, show that if F is finite with q elements, then $F(\alpha)$ has q^n elements. [8]

Question 3

(a) Use Fermat's theorem to compute the remainder when 8^{123} is divided by 13. [6]

(b) For each of the following, find irred(α, Q) and deg(α, Q)

i. $\sqrt{3}+i$

ii. $\sqrt{\frac{1}{5}+\sqrt{7}}$

[6]

(c) Show that if a polynomial $f(x) \in \mathbb{Z}[x]$ is reducible over Q, then it is also reducible over $\mathbb{Z}$. [8]

Question 4

(a) Show that for a field F, the set of all matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}, \quad a_{ij} \in F$$

is a right ideal but not a left ideal of $M_2(F)$. [6]

(b) Let $\varphi_\alpha : \mathbb{Z}_7[x] \to \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homomorphism.

i. $\varphi_2\left(3x^{79} + 5x^{53} + 2x^{43}\right)$

ii. $\varphi_3\left[(x^3+2)(4x^2+3)(x^7+3x^2+1)\right]$

[10]

(c) Show that if D is an integral domain, then $D[x]$ is also an integral domain. [4]

Question 5

(a) Prove that every field is an integral domain. [6]

(b) Which of the following are rings with the usual addition and multiplication?

i. $\{a + \sqrt{2}b : a, b \in \mathbb{Z}\}$

ii. $M_2(\mathbb{R})$ with zero determinant.

[8]

(c) Mark each of the following true or false.

i. Every finite integral domain is a field.

ii. Every division ring is commutative.

iii. $\mathbb{Z}_6$ is not an integral domain.

[6]

Question 6

(a) Classify each of the given $\alpha \in \mathbb{C}$ as algebraic or transcendental over the given field F. If α is algebraic over F, find $\deg(\alpha, F)$.

i. $\alpha = 1 + i, \quad F = \mathbb{Q}$

ii. $\alpha = \sqrt{\pi}, \quad F = Q[\pi]$

iii. $\alpha = \pi^2, \quad F = \mathbb{Q}$

iv. $\alpha = \pi^2, \quad F = Q(\pi^3)$

v. $\alpha = \pi^2, \quad F = Q(\pi)$

[10]

(b) Show that the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is **not** a field. [5]

(c) Find a polynomial of degree greater zero in $\mathbb{Z}_4[x]$ that is a unit. [5]

Question 7

(a) Suppose F is a field, f is an irreducible polynomial over F and g, h are polynomials over F such that f divides gh. Show that either f divides g or f divides h. [10]

(b) Define an ideal N of a ring R. [2]

(c) Find all ideals of $\mathbb{Z}_{10}$ and all maximal ideals of $\mathbb{Z}_{18}$. [8]