UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2009/2010

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set.

(i) What is meant by "(X, d) is a *metric space*?" [2]

(ii) Let d be the Euclidean metric on $\mathbb{R}^2$, and for $x, y \in \mathbb{R}^2$, define

$$d^1(x,y) = \begin{cases} d(x,y) & \text{if } x = sy \text{ or } y = sx \text{ for some } s \in \mathbb{R} \\ d(x,0) + d(0,y) & \text{otherwise.} \end{cases}$$

In other words, $d^1(x, y)$ is the distance you travel from x to y if you can only travel along straight lines through the origin. Verify that d^1 is a metric by checking that conditions $M1. - M3.$ in the definition of a metric space hold for d^1. [3,1,5]

(b) Let (X, d) be a metric space. Given any four points $x, y, z, t \in X$, prove that $|d(x, y) - d(z, t)| \leq |d(x, z) + d(y, t)|$. [4]

(c) A translation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a map given by $T(x) = (x_1 + a, x_2 + b)$ for some fixed point $(a, b) \in \mathbb{R}^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$. Prove that the Euclidian metric d_2 on $\mathbb{R}^2$ is translation invariant, in the sense that for any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we have

$$d_2(T(x), T(y)) = d_2(x, y).$$

[3]

(d) In each of the following cases, state with careful justification whether (X, d) is a metric space:

(i) $X = \mathbb{Q}$ equipped with $d(x, y) = (x - y)^3$;

(ii) $X = \mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ equipped with $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. [1,1]

QUESTION 2

(a) (i) Give the definition of a pseudometric. [2]

(ii) Let X denote the set of all Riemann integrable functions on $[a, b]$. (Recall that a Riemann integrable function on $[a, b]$ need not be continuous in $[a, b]$). For $x, y \in X$, define d by

$$d(x, y) = \int_a^b |x(t) - y(t)| \, dt.$$

Prove that (X, d) is a pseudometric. [5]

(b) Let (X, d) be a metric space, and let A and B be bounded subsets of X. Prove that $A \cup B$ is bounded. Hence, or otherwise, deduce that if x_0 is any point in X, the set A is contained in the closed ball of center x_0 and radius $r = d(x_0, A) + \operatorname{diam}(A)$. [4,1]

(c) Let $A = \{(x_1, x_2) : 0 \le x_1, 0 \le x_2, x_1 + x_2 \le 2\}$ and let $x = (2, 2)$. Find $d(x, A)$ for the Chicago, the London rail, and for the raspberry pickers metrics. [5]

(d) Consider the metric space $(\mathbb{R}, d)$, where d is the usual metric. Prove that the set $\{x\}$ is open as a subspace of $(\mathbb{R}, d)$ but $\{x\}$ is not open as a subset of $\mathbb{R}$ (with the usual metric). [3]

QUESTION 3

(a) Describe the open balls $B(a, 3)$ and the closed balls $B[a, 3]$ in $\mathbb{R}^2$ with respect to the New York metric, where

(i) $a = (0, 0)$, [2,1]

(ii) $a = (2, 3)$. [2,1]

(b) Show that for any two points x and y of a metric space, there exist disjoint open balls such that one is centered at x and the other is centered at y. [5]

(c) Prove that a limit point of a set S in a metric space X is either an interior point or a boundary point of S. [3]

(d) Prove that an isolated point of a set S in a metric space X is a boundary point of S^c. [3]

(e) Prove that a boundary point of a subset S of a metric space X is either a limit point of S or an isolated point of S. [3]

QUESTION 4

(a) Prove that:

(i) Every convergent sequence is a Cauchy sequence; [3]

(ii) If $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are Cauchy sequences in a metric space (X, d), then the sequence $(d(x_n, y_n))_{n\geq 1}$ is convergent in $\mathbb{R}$. [5]

(b) What do you understand by the following:

(i) A nowhere dense metric space; [1]

(ii) An everywhere dense metric space. [1]

(c) (i) Suppose that $\{x_n\}$ converges to x in $C\,[a, b]$ in the uniform metric. Explain what is meant by *pointwise convergence* of a sequence $\{x_n\}$ in $C\,[a, b]$. Show that $\{x_n\}$ converges to x pointwise. [2,3]

(ii) Let x_n in $C\,[0, 1]$ be defined by

$$x_n(t) = \begin{cases} \frac{nt}{n-1} & \text{if } 0 \leq t \leq 1 - \frac{1}{n}, \\ n(1-t) & \text{if } 1 - \frac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that $\{x_n\}$ converges pointwise to the function

$$x(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Deduce that $\{x_n\}$ is not convergent in $C\,[0, 1]$ in the uniform metric. [1,2,2]

QUESTION 5

(a) Let f be the function $f : C[0,1] \longrightarrow \mathbb{R}$ defined for $x \in C[0,1]$ by $f(x) = x(0)$. Show that f is not continuous with respect to the L_1 metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions $x_n(t)$ given by

$$x_n(t) = \begin{cases} (n-1)t & \text{if } 0 \le t \le \frac{1}{n} \\ 1-t & \text{if } \frac{1}{n} \le t \le 1 \end{cases}$$

(**Hint** Sketch the functions $x_n(t)$ and consider their limit in the L_1 metric). [7]

(b) Let (X,d) be a metric space with the metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 3 & \text{if } x \neq y \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X,d) is complete. [3]

(c) (i) Explain what is meant by a *contraction* of a metric space. Show that if $f : [a,b] \longrightarrow [a,b]$ is differentiable, then f is a contraction if and only if there is number $r < 1$ such that $|Df(x)| \le r$ for every $x \in (a,b)$.

(ii) State without proof the *Contraction Mapping Theorem.*

(iii) Show that the mapping $f : [-1,1] \longrightarrow [-1,1]$ defined by $f(x) = \frac{1}{14}(3x^3 - 2x^2 + 9)$ is a contraction, and deduce that there is unique solution to the equation $3x^3 - 2x^2 - 14x + 9 = 0$ in the interval $[-1,1]$. [10]

QUESTION 6

(a) When are two subsets A and B of a metric space said to be separated? [2]

(b) Verify that two nonempty disjoint closed sets in a metric space are separated. [2]

(c) Give two alternate definitions of connectedness of a subset M of a metric space X. [4]

(d) (i) Prove that if X is a connected metric space and $f : X \longrightarrow \mathbb{R}$ is a continuous function, then $f(X)$ is connected.

(ii) Deduce that if $f : [0,1] \longrightarrow [0,1]$ is continuous, then there exists an $x \in [0,1]$ such that $f(x) = x$. [6,6]

QUESTION 7

(a) Let X be a set and let d_1 and d_2 be metrics on X. What is meant by saying that the metrics d_1 and d_2 are *equivalent*? [3]

(b) Suppose that there are positive constants k and K

$$kd_1(x, y) \leq d_2(x, y) \leq Kd_1(x, y)$$

for all $x, y \in X$. Show that d_1 and d_2 are equivalent. [7]

(c) Show that on $\mathbb{R}^2$, the usual (Euclidean) metric and the Chicago metric are equivalent. [4]

(d) Explain what is meant by saying that a metric space X is *connected.* Which of the following subspaces of $\mathbb{R}$ is connected and which is disconnected? Give reasons. (Any theorem about connected subsets of $\mathbb{R}$ that you use should be stated carefully but not proved)

(i) $\mathbb{R} - \mathbb{Q}$,

(ii) $(2, 5) \cup (3, \infty)$,

(iii) $[99, 101)$. [3,2,1]

END OF EXAMINATION