UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2009/2010

BSc. IV

TITLE OF PAPER : FLUID DYNAMICS

COURSE NUMBER : M 455

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :
1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS.
3. NON PROGRAMMABLE CALCULATORS MAY BE USED.

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

USEFUL FORMULAE

The gradient of a function $\psi(r, \theta, z)$ in cylindrical coordinates is

$$\nabla\psi = \frac{\partial\psi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{\hat{\theta}} + \frac{\partial\psi}{\partial z}\underline{\hat{k}}$$

The divergence and curl of the vector field

$$\underline{v} = v_r\underline{\hat{r}} + v_\theta\underline{\hat{\theta}} + v_z\underline{\hat{k}}$$

in cylindrical coordinates are

$$\nabla \cdot \underline{v} = \frac{1}{r}\left\{\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial\theta}(v_\theta) + \frac{\partial}{\partial z}(rv_z)\right\}$$

and

$$\nabla \times \underline{v} = \frac{1}{r}\det\begin{bmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & \underline{\hat{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{bmatrix}$$

The divergence of a vector

$$\underline{v} = v_r\underline{\hat{r}} + v_\lambda\underline{\hat{\lambda}} + v_\theta\underline{\hat{\theta}}$$

in spherical coordinates

$$\nabla \cdot \underline{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta}$$

The convective derivative, Laplacian and strain and shear stress in cylindrical coordinates are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}$$
$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$
$$e_{r\theta} = \frac{1}{2}r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{2r}\frac{\partial V_r}{\partial\theta},\ s_{r\theta} = 2\mu e_{r\theta}$$

Identities

$$\underline{v} \cdot \nabla\underline{v} = \nabla\left(\frac{v^2}{2}\right) - \underline{v} \times \underline{\omega}$$
$$\nabla \times (\nabla \times \underline{a}) = \nabla\nabla \cdot \underline{a} - \nabla^2\underline{a}$$

QUESTION 1

1. (a) Describe a continuum model on the example of the density of the air. [4 marks]

   (b) For the Lagrange method of treating a continuous medium describe the

      i. trajectory of a particle, [2 marks]

      ii. velocity, [2 marks]

      iii. acceleration. [2 marks]

   (c) For the three-dimensional flow

$$\underline{v} = (ay, -ax, f(t)),$$

   where a is a constant, find

      i. particle path, [5 marks]

      ii. streamlines. [5 marks]

QUESTION 2

2. (a) Using the divergence theorem derive the continuity equation in general case. [5 marks]

   (b) For a steady incompressible two-dimensional flow the x component of velocity is given by $u = 2x^2$.

      i. Determine a possible y component, [4 marks]

      ii. How many possible components are there? [1 marks]

   (c) Consider a piston-cylinder apparatus. At one instant when the piston is L_0 away from the closed end of the cylinder, the gas density is uniform at $\rho = \rho_0$ and the piston begins to move away from the closed end at $v = v_0$. The gas velocity is one- dimensional and proportional to the distance from the closed end; it varies linearly from zero at the end to $u = v_0$ at the piston.

      i. Show that $\left.\frac{d\rho}{dt}\right|_{t=0} = \rho_0 \frac{v_0}{L_0}$, [5 marks]

      ii. Find $\rho(t)$. [5 marks]

   Hint: Density ρ is independent of x.

QUESTION 3

3. (a) Derive the convective derivative formula for the density ρ

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{v} \cdot \operatorname{grad} \rho$$

[5 marks]

(b) Consider steady, one-dimensional, incompressible flow along the x-axis through the converging channel

$$\underline{v} = a\left(1 + \frac{x}{L}\right)\hat{\underline{i}}$$

where a and L are constants.

i. Find the acceleration of a particle moving along the x-axis as a function of x, [4 marks]

ii. For the particle located at $x = 0$ at $t = 0$, obtain an expression for its

A. position, x_p, as a function of time, [3 marks]

B. x-component of acceleration as a function of time. [2 marks]

iii. Find the Eulerarian acceleration for a particle moving along the x-axis. [1 marks]

(c) A log is floating in a steady straight river of variable cross section. Find its acceleration

i. in the Eulerarian model, [2 marks]

ii. in the Lagrange model. [3 marks]

<u>QUESTION 4</u>

4. (a) Consider the two-dimensional, incompressible flow.

i. Define the stream function $\psi(x, y)$, [2 marks]

ii. Show that

$$\psi(x, y) = \int_b^y u(x, \eta)d\eta - \int_a^x v(\xi, b)d\xi$$

where a and b are constants. [4 marks]

(b) Consider the radially outwards incompressible flow: $v_r = f(r)$, $v_\theta = 0$. Show that

i. $v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $v_\theta = -\frac{\partial\psi}{\partial r}$, [3 marks]

ii. $\psi = A\theta$, where A is a constant, [3 marks]

iii. $\operatorname{div} \underline{v} = 0$ except at $r = 0$, [1 marks]

iv. $v_r = \frac{q}{2\pi r}$, where q is the source strength. [3 marks]

(c) i. Give the definition of vorticity, and [1 marks]

ii. Find the vorticity for the two-dimensional shear flow $u = \beta y$, $v = 0$. [3 marks]

QUESTION 5

5. (a) i. Explain the following equation

$$dF = \rho \left[\frac{\partial \underline{v}}{\partial t} + u \frac{\partial \underline{v}}{\partial x} + v \frac{\partial \underline{v}}{\partial y} + w \frac{\partial \underline{v}}{\partial z} \right] dx\, dy\, dz$$

[1 marks]

ii. Classify the forces acting on a fluid particle. [1 marks]

(b) i. Define a Newtonian fluid, and [2 marks]

ii. Find the dimension of viscosity. [2 marks]

(c) Write in vector form the Navier-Stokes equation for

i. general case, [2 marks]

ii. statics, [2 marks]

iii. Ideal fluid (invisid model). [2 marks]

iv. Incompressible flow. [2 marks]

(d) An infinite plate is moved over a second plate on a layer of liquid with velocity $v = 0.3$ m/sec. For small gap width $d = 0.3$ mm we assume linear velocity profile. The liquid viscosity is $\mu = 0.65 \times 10^{-3}$ kg/m s and specific gravity $SG = 0.88$. Calculate

i. Kinematic viscosity. [2 marks]

ii. The shear stress on the lower plate. [4 marks]

QUESTION 6

6. (a) A cylindrical container, partly filled with liquid, is rotated at a constant angular velocity ω about the vertical axis. After a short time there is no relative motion, the liquid rotates as a rigid body. Using the Euler equation determine the shape of the free surface if the radius of cylinder is R and the original surface height, in the absence of rotation is h_0. [10 marks]

(b) i. Introducing the characteristic length and velocity, re-write the Navier-Stokes equation in dimensionless form. [5 marks]

ii. Define Reynolds number. [1 marks]

iii. Show that the Reynolds number is dimensionless. [1 marks]

(c) i. Define similar flows. [2 marks]

ii. How is the idea of the similarity of flows used in the design of experimental models? [1 marks]

QUESTION 7

7. (a) Consider stationary viscous incompressible flow between two stationary plates located at $y = 0$ and $y = 1$. Given pressure at $x = 0$ and $x = L$ is P_0 and P_L respectively, $P_0 > P_L$, and the effect of body forces is negligible.

   i. Put $\underline{v} = u(x, y)\hat{\underline{i}}$, simplifiy the Navier-Stokes equations to show that

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

[7 marks]

   ii. Show that

$$P(x) = P_0 - \frac{P_0 - P_L}{L} x,$$

[3 marks]

   iii. Show that

$$u(y) = y(1 - y)\frac{P_0 - P_L}{2\mu L}.$$

[3 marks]

(b) Air flows steadily and at low speed through a horizontal nozzle discharging to the atmosphere. At the nozzle inlet, the area is $0.1\, m^2$. At the nozzle exit, the area is $0.02\, m^2$. The flow is incompressible. and the frictional effects are negligible. Using Bernoulli equation dertermine the pressure required at inlet to produce at outlet speed of $50\, m/s$.

Put $\rho = 1.23\, kg/m^3$. [7 marks]