University of Swaziland

Final Examination, May 2011

BSc II, EEng II, BEd II, BASS II

Title of Paper : Calculus II

Course Number : M212

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Given that

$$f(x,y) = x^2 + xy + y^2 \sin\left(\frac{x}{y}\right)$$

(i) Find f_x, f_y, f_{xx}, f_{xy} and f_{yy}. [5]

(ii) Verify that

$$xf_x + yf_y = 2f \qquad [3]$$

and that

$$x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = 2f. \qquad [4]$$

(b) Using a double integral, find the area of the region bounded by the curves $xy = 2$, $x = 2\sqrt{y}$ and $y = 4$.[8]

Question 2

(a) Find and classify the critical points of the function

$$f(x,y) = y^3 + x^2 - 8xy + 3x + 6y. \qquad [10]$$

(b) Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x,y,z) = xyz$$

subject to

$$x^2 + y^2 + z^2 = 1. \qquad [10]$$

Question 3

(a) Consider the cardioid

$$r = 1 - \cos\theta.$$

(i) Sketch the cardioid.

(ii) Find the legth of the cardioid

[12]

(b) Find an equation in polar co-ordinates for each of the following curves

(i) $2x + 3y = 3$

(ii) $x^2 - 2x + y^2 = 0$

[8]

Question 4

Evaluate the following integral

(a) $\int_0^1 \int_0^{\sqrt{x-x^2}} y^2 \mathrm{d}y\mathrm{d}x$ [10]

(b) $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-x^2}} x^3yz \, \mathrm{d}x\mathrm{d}y\mathrm{d}z$ [10]

Question 5

(a) Suppose that $z = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$. Prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2. \quad [10]$$

(b) Find the directional derivative of

$$z = f(x, y) = x^3 e^y + xz$$

in the direction of the vector from $P_1(4, 0, 16)$ to $P_2(-2, 1, 4)$. [10]

Question 6

(a) Find the volume under the surface

$$z = x^4 y^4$$

and over the circle $x^2 + y^2 = 1$. [12]

(b)

(i) Sketch the graph of the curve

$$r = 1 - \sin\theta.$$

(ii) Find the area of the region enclosed by the curve in (i).

[8]

Question 7

(a) Find the equation of the tangent surface $xyz^3+yz^2 = 4$ at the point $(1, 2, 1)$.

(b) Find the equation of the plane through the 3 points $P(1, 2, 3)$, $Q(-2, 0, 4)$ and $R(5, 2, -1)$.

(c) Evaluate

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x\mathrm{d}y$$

where R is the region bounded by the lines $y = x$, $y = -2$ and $x = 0$.