UNIVERSITY OF SWAZILAND

51

FINAL EXAMINATION 2010/2011

B.A.S.S. /BEd. /BEng. /BSc. II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Solve the linear system

$$\begin{array}{rcrcrcr} 2x & & & - & z & = & 4 \\ x & + & y & + & z & = & 6 \\ 3x & + & y & & & = & 10 \end{array}$$

using Gaussian elimination. [8 marks]

(b) Express the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

as a product of elementary matrices. [12 marks]

QUESTION 2

2. (a) Given the linear system

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 2 \\ 3x_1 & + & 3x_2 & + & \alpha x_3 & = & \beta \\ -x_1 & + & 3x_2 & + & 2x_3 & = & -8 \end{array}$$

find values of α and β for which the system has;

i. no solutions,

ii. a unique solution,

iii. infinitely many solutions. [10 marks]

(b) Prove that if square matrices A and B are invertible, then BA is invertible and $(BA)^{-1} = A^{-1}B^{-1}$. [5 marks]

(c) Let A be a 2×2 matrix and let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If $AB = BA$, then show that $A = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}$ for some numbers α and β. [5 marks]

QUESTION 3

3. (a) Write down any 4 axioms for a vector space. [4 marks]

(b) Explain precisely what it means to say that a non-empty set W is a subspace of a vector space V. [4 marks]

(c) Determine whether or not the following subsets are subspaces. Justify your answers.

i. $W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 + 2x_n = 0\}$ in $\mathbb{R}^n$. [4 marks]

ii. $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : cd = 0 \right\}$ in the set M_{22} of all of all 2×2 matrices. [4 marks]

iii. $W = \{p(x) \in P_3 | p(0) = 0\}$ in the set P_3 of all polynomials of degree at most 3. [4 marks]

QUESTION 4

4. (a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in a vector space V. Explain precisely what is meant by each of the following statements.

i. S spans V, [2 marks]

ii. S is linearly independent in V, [2 marks]

iii. S is a basis for V. [2 marks]

(b) Determine whether or not the set

$$\{(1, 2, 0), (1, 2, 1), (2, 4, 3)\}$$

spans $\mathbb{R}^3$. [4 marks]

(c) Determine whether or not the set

$$\{(1, -1, 1), (2, 0, 1), (7, -3, 5)\}$$

is linearly independent in $\mathbb{R}^3$. [4 marks]

(d) i. Define the nullspace of an $m \times n$ matrix. [2 marks]

ii. Find a basis for the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

[4 marks]

QUESTION 5

5. Let U and V be vector spaces.

   (a) What does it mean to say that T is a linear transformation from U to V? [2 marks]

   (b) Are the following linear transformations? Justify your answers.

   i. $T : P_1 \to P_2$; $T(a_0 + a_1x) = \int(a_0 + a_1x)dx$. [4 marks]

   ii. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x_1, x_2) = (x_1 + x_2, x_1x_2)$. [4 marks]

   iii. $T : M_{22} \to \mathbb{R}$; $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$. [4 marks]

   (c) i. Let U and V be vectorspaces, and let $T : U \to V$ be a linear transformation.

   A. Define the image of T. [2 marks]

   B. Define the kernel of T. [2 marks]

   ii. If $T : U \to U$ is defined by $T(p(x)) = p''(x)$, then describe the kernel of T. [2 marks]

QUESTION 6

6. (a) Let P_2 denote the set of all polynomials of degree at most 2.

   i. Show that

$$\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$$

   defines an inner product on the vector space P_2. [6 marks]

   ii. Define the norm $\|p(x)\|$ of a polynomial $p(x) \in P_2$ with respect to the inner product in 6(a)i above. [2 marks]

   iii. Compute $\|p(x)\|$ when $p(x) = \sqrt{3}x$. [2 marks]

   (b) If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

   then find the eigenvalues and eigenvectors of A. [10 marks]

QUESTION 7

7. (a) State the Cayley-Hamilton theorem. [2 marks]

(b) Verify the Cayley-Hamilton theorem with the matrix

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

[8 marks]

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$$

Find bases for the image and kernel of T. [10 marks]