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UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2010/2011

B.A.S.S. /BEd. /BEng. /BSc. II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Solve the linear system

$$\begin{aligned} x + y + z &= 6 \\ 2x \quad\quad - z &= 4 \\ 3x + y \quad\quad &= 10 \end{aligned}$$

using Gaussian-Jordan elimination. [8 marks]

(b) Express the matrix

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

as a product of elementary matrices. [12 marks]

QUESTION 2

2. (a) Given the linear system

$$\begin{aligned} 3x_1 + 3x_2 + \beta x_3 &= \alpha \\ x_1 \quad\quad + 3x_3 &= 2 \\ -x_1 + 3x_2 + 2x_3 &= -8 \end{aligned}$$

find values of α and β for which the system has;

i. no solutions,

ii. a unique solution,

iii. infinitely many solutions. [10 marks]

(b) Prove that if a square matrices A is invertible, then A^2 is invertible and $(A^2)^{-1} = (A^{-1})^2$. [5 marks]

(c) Let A be a 2×2 matrix and let $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. If $AB = BA$, then show that $A = \begin{bmatrix} \gamma + \delta & 0 \\ \gamma & \gamma + \beta \end{bmatrix}$ for some numbers α and β. [5 marks]

QUESTION 3

3. (a) Write down any 5 axioms for a vector space. [5 marks]

   (b) State the **subspace test** for determining whether or not a non-empty set W is a subspace of a vector space V. [3 marks]

   (c) Determine whether or not the following subsets are subspaces. Justify your answers.

      i. $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 1 \right\}$ in the set M_{22} of all of all 2×2 matrices. [4 marks]

      ii. $W = \{p(x) \in P_2 | p'(0) = 0\}$ in the set P_2 of all polynomials of degree at most 2. [4 marks]

      iii. $W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 + x_n > 0\}$ in $\mathbb{R}^n$. [4 marks]

QUESTION 4

4. (a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors in a vector space V. Explain precisely what is meant by each of the following statements.

      i. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ span V, [2 marks]

      ii. S is linearly independent in V, [2 marks]

      iii. S is a basis for V. [2 marks]

   (b) Determine whether or not the set

   $$\{(1,0,0), (0,1,0), (1,0,1)\}$$

   spans $\mathbb{R}^3$. [4 marks]

   (c) Determine whether or not the set

   $$\{(1,2,3), (4,5,6), (7,8,9)\}$$

   is linearly independent in $\mathbb{R}^3$. [4 marks]

   (d) i. Define the column space Col A of an $m \times n$ matrix A. [2 marks]

      ii. Given the set

      $$W = \left\{ \begin{bmatrix} a+b+c \\ a+b \\ a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

      determine a suitable matrix A such that $W = \text{Col } A$. [4 marks]

QUESTION 5

5. Let U and V be vector spaces.

   (a) What does it mean to say that T is a linear map from U to V? [2 marks]

   (b) Are the following linear maps? Justify your answers.

   i. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x_1, x_2) = (x_1^2, x_1^2)$. [4 marks]

   ii. $T : P_2 \to P_1$; $T(a_0 + a_1x + a_2x^2) = \dfrac{d}{dx}(a_0 + a_1x + a_2x^2)$. [4 marks]

   iii. $T : M_{22} \to \mathbb{R}$; $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b + c$. [4 marks]

   (c) i. Let U and V be vectorspaces, and let $T : U \to V$ be a linear map.

   A. Define the kernel of T. [2 marks]

   B. Define the nullity of T. [2 marks]

   ii. If $T : U \to U$ is defined by $T(p(x)) = p''(x)$, then what is the nullity of T? [2 marks]

QUESTION 6

6. (a) Let M_{22} denote the set of all 2×2 matrices in the usual manner. The trace tr A of a square matrix A is the sum of all its diagonal entries.

   i. Show that

   $$\langle A, B \rangle = \mathrm{tr}\,(B^T A)$$

   defines an inner product on the vector space M_{22}. [6 marks]

   ii. Define the norm $\|A\|$ of a matrix $A \in M_{22}$ with respect to the inner product in 6(a)i above. [2 marks]

   iii. Compute $\|A\|$ when $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. [2 marks]

   (b) If

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

   then find the eigenvalues and eigenvectors of A. [10 marks]

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QUESTION 7

7. (a) State the Cayley-Hamilton theorem and verify it with the matrix

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

[10 marks]

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map defined by

$$T(x_1, x_2, x_3) = (x_1, x_2 - x_3)$$

Find bases for the image and kernel of T. [10 marks]