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UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2010/11

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

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QUESTION 1

1.1 State the difference between a valid argument and an invalid argument. [4]

1.2 A detective has interviewed four witnesses to a crime, namely a butler, a cook, a gardener and a handyman. From the stories of the witnesses, the detective has concluded that if the butler is telling the truth, then so is the cook; the cook and the gardener cannot both be telling the truth, the gardener and the handyman are not both lying; and if the handyman is telling the truth, then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning. [6]

1.3 A certain island has two kinds of inhabitants; knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described below. If you you cannot determine what these people are, can you draw any conclusions?

(a) A says, "At least one of us is a knave," and B says nothing.

(b) A says, "I am a knave or B is a knight," and B says nothing.

(c) A says, "We are both knaves," and B says nothing. [10]

QUESTION 2

2.1 Determine the following sets:

(a) $\{n \in \mathbb{N} : \exists m \in \mathbb{N} \text{ with } m \leq n\}$; [3]

(b) $\{n \in \mathbb{N} : \forall m \in \mathbb{N} \text{ we have } m \leq n\}$. [2]

2.2 Let a be an algebraic number and r a rational number. Show that ra is an algebraic number. [5]

2.3 Suppose you want to show that $A \Rightarrow B$ is **false**. How should you do this? What should you try to show about the truth of A and B? [2]

2.4 Apply your answer of part (a) to show that the statement "If x is a real number that satisfies $-3x^2 + 2x + 8 = 0$, then $x > 0$" is false. [3]

2.5 Write the negation of the statement: "The real-valued function f of one variable is *continuous at the point* x if and only if for every real number $\varepsilon > 0$, there is a real number $\delta > 0$ such that, for all real numbers y with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$." [5]

QUESTION 3

3.1 Show that $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ is a tautology. [8]

3.2 Use truth table analysis to show that:

(a) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$; [6]

(b) $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$. [6]

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QUESTION 4

4.1 Prove that in any set of $n + 1$ pairwise distinct integers, there must be two whose difference is divisible by n. [7]

4.2 Prove, by the contrapositive method, that if no angle of a quadrilateral $RSTU$ is obtuse, then the quadrilateral $RSTU$ is a rectangle. [6]

4.3 (a) Show that if r is a nonzero rational number, then $r\sqrt{7}$ is an irrational number. [4]

(b) Using the result in part (a), or otherwise, show that $\sqrt{28}$ is irrational. [3]

QUESTION 5

5.1 Prove that $A \subseteq B \Leftrightarrow A \cap B = A$. [5]

5.2 Using the fact that an implication is equivalent to its contrapositive, prove that, for subsets of a universal set $\mathscr{U}$, $A \subseteq B \Leftrightarrow B^c \subseteq A^c$. [5]

5.3 Let p_1 and p_2 be distinct prime numbers. Prove that the real numbers $\sqrt{p_1}+\sqrt{p_2}$ and $\sqrt{p_1} - \sqrt{p_2}$ are irrational. [10]

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QUESTION 6

6.1 Using truth tables, analyze the following argument and state whether it is valid or invalid.

> "Rock festivals are designed for adolescents. Nothing designed for adolescents has any cultural value. Therefore rock festivals have no cultural value. "

[6]

6.2 (a) Define an equivalence relation. [2]

(b) Let $\mathbb{Z}$ be the set of all integers and let

$$\mathscr{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod{3}\}$$

be a relation on $\mathbb{Z}$. Show that $\mathscr{R}$ is an equivalence relation. What are the equivalence classes of $\mathscr{R}$? [12]

QUESTION 7

7.1 (a) Define a square-free natural number. [2]

(b) Let b and m be two natural numbers such that b is square-free and m^2 is divisible by b. Prove that m is also divisible by b. [10]

7.2 Let S and T be finite sets, and let $f : S \longrightarrow T$ be a function. Prove that:

(a) If f is onto, then $|S| \geq |T|$; [3]

(b) If f is one-to-one, then $S| \leq |T|$; [3]

(c) If f is a bijection, then $|S| = |T|$. [2]

END OF EXAMINATION