UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2010/2011

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	NUMERICAL ANALYSIS I
COURSE NUMBER	:	M 311
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Consider the iterative schemes:

$$x_{n+1} = \frac{1}{4}\left(x_n^2 + \frac{6}{x_n}\right) \quad \text{and} \quad x_{n+1} = 4 - \frac{6}{x_n^2}, \quad n \geq 0$$

for solving the non-linear equation $x^3 = 4x^2 - 6$ in the interval [3,4]. Which one is suitable for approximating the solution of the given equation? [4 Marks]

(b) Determine the decimal numbers that has the following 32-bit floating point representation

1 1000 0100 101 0000 1111 0000 0000 0000

[6 Marks]

(c) Determine the machine representation in single precision on a 32-bit word length computer (Marc-32) for the number -8×2^{-24} [6 Marks]

(d) Show that $f(x) = x^3 + 4x^2 - 10$ has exactly one root in [1,2]. [4 Marks]

QUESTION 2

2. (a) Find the roots of the following quadratic equation (as accurately as possible) using eight digits and rounding

$$x^2 - 100000x + 1 = 0$$

[6 Marks]

(b) Consider the iterative scheme

$$x_{k+1} = (\alpha + 1)x_k - x_k^2$$

i. Find the fixed points of the scheme. [2 Marks]

ii. Show that the interval where this scheme is guaranteed to converge is

$$1 + \frac{\alpha}{2} < x < \frac{\alpha}{2}$$

[4 Marks]

(c) The quadrature formula

$$\int_{-1}^{1} f(x)\, dx \approx c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact for all polynomials of degree less than or equal to 2.

Determine c_0, c_1 and c_2. [8 Marks]

QUESTION 3

3. (a) Given the following data

x	-1	-0.5	0	0.5	1.0	1.5
$f(x)$	0.130	0.146	0.169	0.202	0.249	0.314

i. Construct a forward divided difference table for the above tabulated data. [5 marks]

ii. Use Newton interpolation formula to approximate $f(-0.3)$. [5 marks]

(b) Construct a divided difference table for the following function values

$$f(-1) = 1, \quad f(0) = 1, \quad f(3) = 181, \quad f(-2) = -39, \quad f(4) = 801$$

and use it to obtain a polynomial that interpolates the function values. [10 marks]

QUESTION 4

4. (a) Let $p_2(x)$ be the quadratic polynomial interpolating $f(x)$ at $(0, f(0))$, $(h, f(h))$ and $(2h, f(2h))$.

i. Write down the Lagrange representation of $p_2(x)$. [5 marks]

ii. By integrating $p_2(x)$ between 0 and $3h$, derive the following numerical integration rule that approximates $I = \int_0^{3h} f(x)\,dx$; that is, show that

$$I \approx \frac{3h}{4}\left[f(0) + 3f(2h)\right].$$

[7 marks]

(b) Given the function $f(h) = \sqrt{9-h} - 3$

(i) find a suitable function $g(h)$ that has been reformulated to be algebraically equivalent to $f(h)$ with the aim of avoiding loss of significance error. [2 marks]

(ii) Compare the results of calculating $f(0.0001)$ and $g(0.0001)$ using **six digits** and **chopping**. [4 marks]

QUESTION 5

5. (a) Find the LU factorisation of the matrix A in which the diagonal elements of L are 1 for

$$A = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 2 & 2 & 0 \\ 0 & 2 & 8 & -6 \\ 0 & 0 & -6 & 10 \end{bmatrix}$$

[8 Marks]

(b) Use the LU factorisation in (a) to find x such that

$$Ax = \begin{bmatrix} 2 \\ 0 \\ -8 \\ 16 \end{bmatrix}$$

[6 Marks]

(c) Use the two-point Gaussian Quadrature rule,

$$\int_{-1}^{1} f(x)\, dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right),$$

to approximate the integral

$$\int_{0}^{1} x^2 e^{-x}\, dx$$

and compare your result against the exact value of the integral. [6 marks]

QUESTION 6

6. (a) Suppose we know the following values of a function f:

$$f(0) = 1, \quad f(0.5) = 2.5, \quad f(1) = 2, \quad f(0.25) = \alpha, \quad f(0.75) = \alpha$$

Find α if the composite trapezoidal rule with $n = 4$ gives

$$\int_0^1 f(x)\, dx = 1.75$$

[5 marks]

(b) Consider the integral $\int_0^1 \sin\left(\frac{\pi x^2}{2}\right)\, dx$. Suppose we wish to integrate it numerically with an error of magnitude less than 10^{-3}. What width h is needed if we wish to use the composite Trapezoid rule? [5 marks]

(c) Use LU decomposition to solve the following linear system

$$\begin{aligned} x + 2y + 3z &= 4 \\ x - y + 6z &= -1 \\ 2x + y &= 0 \end{aligned}$$

[10 marks]

QUESTION 7

7. Let $f(x) = x - e^{-x}$

(a) Show that $f(x)$ has exactly one root in $[0, 1]$. [5 marks]

(b) Compute an approximation to the root by taking 4 steps of the bisection method. [6 marks]

(c) How many iterations would be required to locate this zero to a tolerance of 10^{-5}? [3 marks]

(d) Compute an approximation to the root by taking 3 steps of the Newton's method starting with $x_0 = 0.5$. [6 marks]