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UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2010/2011

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

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QUESTION 1

(a) Evaluate:

(i) $\Gamma(6.8)$, given that $\Gamma(1.8) = 0.9314$, [2]

(ii) $\int_0^\infty x^m e^{-ax^n} dx$, where m and n are positive integers. [8]

(b) Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}, \text{ where } m, n > 0.$$ [10]

QUESTION 2

(a) Find a parametrization of the cylinder

$$x^2 + (y-a)^2 = a^2 \qquad 0 \leq z \leq 5,$$

where a is a constant. [6]

(b) Find the distance from the plane $x + 2y + 6z = 10$ to the plane $x + 2y + 6z = 20$. [7]

(c) Find the angle between the planes $x + y = 1$ and $2x + y - 2z = 2$. [7]

QUESTION 3

(a) Express the following in cylindrical coordinates:

(i) gradϕ;

(ii) div$\mathbf{F}$;

(iii) the volume element dV, and

(iv) the Jacobian. [10]

(b) Let D be the region in the xyz-space defined by the inequalities

$$1 \leq x \leq 2, \qquad 0 \leq xy \leq 2, \qquad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2 y + 3xyz)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

by applying the transformation

$$u = x, \qquad v = xy, \qquad w = 3z$$

and integrating over the appropriate region G in the uvw-space. [10]

QUESTION 4

(a) Let $\mathbf{F} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{\mathbf{k}}$ be a vector field.

(i) Show that $\mathbf{F}$ is irrotational. [3]

(ii) Find div curl $\mathbf{F}$. [3]

(b) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (4,0) following a cubic curve. Find the equation of this curve if the track is continuous, smooth, and has continuous curvature. [10]

(c) Find parametric equations for the line through $(1, -6, 1)$ perpendicular to the plane $x + 2y + 2z = 13$. [4]

QUESTION 5

(a) Determine the directional derivative of $\phi(x, y) = \ln \sqrt{x^2 + y^2}$ at the point $(1, 0)$ in the direction of $\dfrac{2\hat{\imath} + 2\hat{\jmath}}{2\sqrt{2}}$. [6]

(b) Find a unit normal to the surface $2x^2 + 4yz - 5z^2 = -100$ at $P(2, -2, 3)$. [6]

(c) Find a parametrization of the first-octant portion of the cone $z = \dfrac{\sqrt{x^2 + y^2}}{2}$ between the planes $z = 0$ and $z = 3$. [8]

QUESTION 6

(a) If $\mathbf{A} = (3x^2 - 6yz)\hat{\mathbf{i}} + (2y + 3xz)\hat{\mathbf{j}} + (1 - 4xyz^2)\hat{\mathbf{k}}$, evaluate $\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

(i) $x = t,\ y = t^2,\ z = t^3$;

(ii) the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(1,1,1)$;

(iii) the straight line joining $(0,0,0)$ and $(1,1,1)$. [12]

(b) Verify Green's theorem in the plane for

$$\oint_C [2x\mathrm{d}x - 3y\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [8]

QUESTION 7

(a) By any method, find the integral of $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$. [10]

(b) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\mathbf{i}} + (2y + x^2z)\hat{\mathbf{j}} + (4x^2y^3)\hat{\mathbf{k}}$ across the boundary of the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$. [10]

END OF EXAMINATION