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University of Swaziland

Final Examination, May 2011

BSc III, Bass III, BEd III

Title of Paper : Complex Anlysis

Course Number : M313

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

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(a) (i) Prove that $\text{Im}(iz) = \text{Re}\, z$. [2]

(ii) Represent in exponential form

$$1 - \cos\alpha + i\sin\alpha.$$

[5]

(b) In the complex plane, define

(i) deleted neighbourhood of a point, [1]

(ii) boundary of a set S, [1]

(iii) closure of a set S, [1]

(c) (i) Sketch the following sets:

$$|2z+3| > 6; \qquad \text{Im}\, z < 2; \qquad 0 \le \arg z \le \pi/3.$$

[2]

(ii) Which sets in (i) are domains? [2]

(iii) Which sets in (i) are bounded? [2]

(d) Construct the line

$$\text{Im}\,\frac{1}{z-i} = \frac{1}{2}.$$

[4]

QUESTION 2

(a) Find the region into which a transformation $w = f(z)$ maps a region D if

(i) $w = 2 + z^2$, D is a triangle bounded by the lines $x = 1, y = x$ and $y = 0$; [3]

(ii) $w = \dfrac{1}{z}$, D is the half ring

$$1 \le |z| \le 2, \quad \text{Im}\, z \ge 0.$$

Hint: pass to polars. [3]

(b) Find the limits. Explain.

(i) $\displaystyle\lim_{z\to 1} \frac{1}{(z-1)^2}$; [2]

(ii) $\displaystyle\lim_{z\to\infty} \frac{2z+3i}{z+2}$. [2]

(c) Define a function $f(z)$ continuous at a point z_0. [2]

(d) Using just the definition of the derivative, find $f'(z)$ for the following $f(z)$:

(i) $f(z) = |z|^2$. Use $z\bar{z} = |z|^2$. [6]

(ii) $f(z) = \text{Im}\, z$. [2]

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QUESTION 3

(a) Using Cauchy-Riemann equations,

(i) state a sufficient condition theorem for existence of $f'(z_0)$, and thus

(ii) check if there are derivatives of $f(z) = z^2 + 3z + 2$, and $g(z) = e^x(\cos y + i \sin y)$, $z = x + iy$. [1,6]

(b) Use Cauchy-Riemann equations to show that $f'(z)$ does not exist if

$$f(z) = 2x + ixy^2, \quad z = x + iy.$$

[3]

(c) Use the results from Q3 (a) (i) to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$ when

$$f(z) = 3x + y + i(3y - x), \quad z = x + iy.$$

[4]

(d) Derive the Cauchy-Riemann equations in polar coordinates. [6]

QUESTION 4

(a) Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic, $z = x + iy$, in domain D, then u nad v are harmonic in D. [5]

(b) Consider $f(z) = \dfrac{1}{z}$

(i) Is $f(z)$ analytic? Explain. [2]

(ii) Find out if there are any singular points. Explain. [2]

(c) Given $u(x, y) = e^x [x \cos y - y \sin y]$

(i) Find $f(z)$ as an explicit function of z, where the real part of $f(z)$ is $u(x, y)$. [8]

(ii) Find $f'(z)$. [3]

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QUESTION 5

(a) Evaluate the following integral

$$I = \int_C z^{1/2}\, dz,$$

where C is a positively oriented half cirlce $C = \{z : |z| = 3,\ 0 \le \arg z \le \pi\}$. [6]

(b) (i) Define a simple closed curve. [1]

(ii) State [1]

(iii) derive the Cauchy formula for continuous $f'(z)$. [6]

Hint: Apply Green's Theorem $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)dxdy.$

(c) Evaluate

$$\int_C \frac{dz}{z - z_0},$$

where C is the positively oriented circle $|z - z_0| = R$. [6]

QUESTION 6

(a) Expand $\dfrac{1}{z^2 + 3z}$ into simple fractions, and thus show that

$$\int_C \frac{dz}{z^2 + 3z} = \frac{2\pi i}{3}.$$

[6]

(b) State the Laurent series theorem. [3]

(c) Find the Maclaurin series expansion for

$$f(z) = z^2 e^{3z}.$$

[3]

(d) Give the Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1 - z)}$$

valid in

(i) $0 < |z| < 1$; [4]

(ii) $1 < |z| < \infty$. [4]

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QUESTION 7

(a) Using the Laurent series expansion in the neighbourhood of isolated singular point, derive the formula

$$\int_C f(z)\,dz = 2\pi i b_1.$$

[3]

(b) Find the residue at $z = 0$ for the function

$$f(z) = \frac{1}{4z - z^2}.$$

[3]

Hint: $\displaystyle\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, for $|z| < 1$.

(c) Consider the function

$$f(z) = \frac{1 - \cosh z}{z^3}.$$

(i) Show that $\cosh z = \cos(iz)$. [3]

(ii) Show that a singular point is a pole and find its order. [3]

(iii) Find the residue. [1]

(d) Using the residue theorem, evaluate

$$\int_0^{\infty} \frac{\cos x}{x^2 + a^2}\,dx, \quad a > 0.$$

[7]

END OF EXAMINATION PAPER