103

# University of Swaziland

## Supplementary Examination, July 2011

## BSc III, Bass III, BEd III

**Title of Paper** : Complex Anlysis

**Course Number** : M313

**Time Allowed** : Three (3) hours

**Instructions** :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

## QUESTION 1

(a) (i) Prove that $\operatorname{Re}(iz) = -\operatorname{Im} z$. [2]

(ii) Solve $z^6 + 64 = 0$. [5]

(b) In the complex plane, define

(i) $\epsilon$-neighbourhood of a point $z_0$, [1]

(ii) connected set, [1]

(iii) accumulation point. [1]

(c) (i) Sketch the following sets:

$$|z - 3 + i| \leq 1; \qquad \operatorname{Im} z > 2; \qquad \pi/6 < \arg z \leq \pi/3.$$

[2]

(ii) Which sets in (i) are domains? [2]

(iii) Which sets in (i) are bounded? [2]

(d) Construct the line

$$\operatorname{Re} \frac{1}{z+2} = \frac{1}{4}.$$

[4]

## QUESTION 2

(a) Find the region into which a transformation $w = f(z)$ maps a region $D$ if

(i) $w = e^z$, $D$ is a rectangle bounded by the lines $x = 0, x = 2, y = 0$ and $y = 1$; [3]

(ii) $w = z^2$, $D$ is the closed triangular region formed by the lines $y = \pm x$ and $x = 1$. [3]

(b) Find the limits. Explain.

(i) $\lim_{z \to i} \dfrac{iz + 2i}{z - i}$; [2]

(ii) $\lim_{z \to \infty} \dfrac{z^2 + 1}{z - 1}$. [2]

(c) Define a function $f(z)$ uniformly continuous in a region $R$. [2]

(d) Using just the definition of the derivative, find $f'(z)$ for the following functions:

(i) $f(z) = |z|^2$. Use $|z|^2 = z\bar{z}$. [6]

(ii) $f(z) = \bar{z}$. [2]

## QUESTION 3

(a) (i) State and

(ii) Prove the necessary conditions for the existence of $f'(z)$. [1,5]

(b) Use Cauchy-Riemann equations to show that $f'(z)$ does not exist if

$$f(z) = e^x e^{-iy}, \quad z = x + iy.$$

[4]

(c) Use the sufficient conditions theorem to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$ when

$$f(z) = z^3.$$

[4]

(d) (i) Write the Cauchy-Riemann equations in polar coordinates. [1]

(ii) Let $f(z) = \frac{1}{z}$. Pass to the polar coordinates and use results from (i) to check if $f'(z)$ exists, and if yes, find it. [5]

## QUESTION 4

(a) Prove that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is analytic in domain $D$, if and only if $v$ is harmonic in conjugate of $u$. [6]

(b) Consider $f(z) = \frac{1}{z}$

(i) Is $f(z)$ analytic? Explain. [2]

(ii) Find out if there are any singular points. Explain. [2]

(c) Given $u(x, y) = y^3 - 3x^2y$,

(i) Find $v(x, y)$, the harmonic conjugate of $u(x, y)$. [6]

(ii) Is $u(x, y)$ the harmonic conjugate of $v$? Explain. [4]

## QUESTION 5

(a) Evaluate $\int_C \bar{z}\, dz$, where $C$ is the right hand half cirlce $|z| = 2$, from $z = -2$ to $z = 2i$. [6]

(b) (i) Define a simple (Jordan) arc. [1]

(ii) State, [1]

(iii) and prove the Cauchy-Goursat (CG) theorem for multiply connected domains. [6]
**Hint:** Apply CG Theorem for simply connected domains.

(c) Apply the Cauchy integral formula to evaluate

$$\int_C \frac{z}{(16 - z^2)(z + i)},$$

where $C$ is the positively oriented circle $|z| = 2$. [6]

## QUESTION 6

(a) Apply the Cauchy integral formula to show that

$$\int_C \frac{dz}{z^2 + 4} = \frac{\pi}{2},$$

where $C$ is the positively oriented circle $|z - i| = 2$. [6]

(b) (i) State the Taylor series theorem and thus

(ii) expand $\frac{1}{1 - z}$ in a Maclaurin series for $|z| < 1$. [2,3]

(c) (i) Expand

$$f(z) = \frac{1}{(z - 1)^2(z - 3)}$$

in a Laurent series in powers of $z - 1$ valid for $0 < |z - 1| < 2$. [8]

(ii) What is the principal part of the series in (i)? [1]

## QUESTION 7

(a) Find the residue at $z = 0$ for the following function

$$f(z) = z \cos \frac{1}{z}.$$

[3]

(b) For the function $f(z) = \exp\left(\frac{1}{z^2}\right)$,

(i) Find the residue, and thus

(ii) evaluate the integral

$$\int_C \exp\left(\frac{1}{z}\right)\, dz,$$

where $C$ is the positively oriented circle $|z| = 2$.

[3,1]

(c) (i) State, [1]

(ii) and prove the residue theorem. [4]

(d) Apply the residue theorem to evaluate

$$\int_C \frac{dz}{z^3(z+4)},$$

where $C$ is the positively oriented circle

(i) $|z| = 2$; [4]

(ii) $|z + 2| = 3$. [4]

END OF EXAMINATION PAPER