UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2010/2011

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let S be a set of real numbers. Explain precisely what is meant by each of the following statements.

   i. A real number α is an upper bound for S. [2 marks]

   ii. A real number β is a supremum for S. [2 marks]

   (b) State the supremum property for set $\mathbb{R}$ of all real numbers. [2 marks]

   (c) Does each of the following sets have a supremum? Justify your answer.

   i. $S := \{x \in \mathbb{R} : |x| + |x+1| > 2\}$. [4 marks]

   ii. $S := \{x \in \mathbb{R} : |x| = |x+1|\}$. [4 marks]

   (d) Prove that the sum of a rational number and an irrational number is always irrational. [3 marks]

   (e) Let S be a non-empty set of real numbers, and let $\alpha = \sup S$. Also, let $T := \{ax : a > 0,\, x \in S\}$. Show that $\sup T = a\alpha$. [3 marks]

QUESTION 2

2. (a) Let (x_n) be a sequence of real numbers. Explain precisely what is meant by each of the following statements.

   i. A real number l is a limit of the sequence (x_n). [3 marks]

   ii. The sequence (x_n) converges. [1 marks]

   (b) Prove that any sequence (x_n) of real numbers has a unique limit. [4 marks]

   (c) Show that if x is the limit of the sequence (x_n) of real numbers then $|x|$ is the limit of the sequence $(|x_n|)$. [4 marks]

   (d) i. Explain precisely what it means to say that "*a sequence (x_n) of real numbers is Cauchy*". [3 marks]

   ii. State the Cauchy convergence criterion for a sequence of real numbers. [2 marks]

   iii. Use the Cauchy convergence criterion to prove that the sequence $\left(\dfrac{1}{n}\right)$ is convergent. [3 marks]

QUESTION 3

3. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, and let $c \in (a, b)$.
   - i. Explain precisely what it means to say that f is continuous at c. [2 marks]
   - ii. Suppose that both f and g are continuous at c, then prove each of the following statements.
     - A. The difference $f - g$ is continuous at c. [4 marks]
     - B. The scalar multiple αf is continuous at c. [4 marks]
   - iii. Give examples of functions $f, g : [-1, 1] \to \mathbb{R}$ such that $f - g$ is continuous and yet neither f nor g is continuous. [2 marks]

   (b)
   - i. State the Intermediate value theorem. [2 marks]
   - ii. Show that the equation $-x = \cos x$ has a solution in the interval $[-\pi/2, 0]$. [3 marks]

   (c) Is the following statement true or false? Justify your answer.
   If a function $f : (0, 1) \to \mathbb{R}$ is continuous then f is bounded. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$ be a function.
   - i. Explain what is meant by saying that f is differentiable at $c \in (a, b)$. [2 marks]
   - ii. Show if f is differentiable at point c, then f is continuous at $c \in (a, b)$. [4 marks]
   - iii. Give an example of a function $f : (-1, 0) \to \mathbb{R}$ that is continuous at some point $c \in (-1, 0)$ and yet f not differentiable at c. [2 marks]

   (b)
   - i. State the Mean value theorem. [2 marks]
   - ii. Use the Mean value theorem to prove each of the following statements.
     - A. $|\sin x| \leq |x|, \ \forall x \in \mathbb{R}$. [5 marks]
     - B. $a - b \leq \sin b - \sin a \leq b - a$ for $a < b$. [5 marks]

QUESTION 5

5. (a) Let $\sum a_n$ be a series in $\mathbb{R}$. Then, explain the following statements.
   i. The k-th partial sum. [2 marks]
   ii. $\sum a_n$ converges. [2 marks]
   iii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) Show that if $\sum a_n$ converges, then $\lim(a_n) = 0$. [3 marks]

   (c) Give an example of series $\sum a_n$ such that $\lim(a_n) = 0$ and yet $\sum a_n$ does not converge. [2 marks]

   (d) State the Cauchy convergence criterion for series. [2 marks]

   (e) Prove that if $\sum a_n$ is absolutely convergent, then $\sum a_n$ converges. [4 marks]

   (f) Determine whether the series
   $$\sum \frac{(-1)^{n+1}}{n}$$
   converges or not. State any theorems used. [4 marks]

QUESTION 6

6. (a) State Riemann's criterion for integrability. [2 marks]

   (b) Prove that if $f : [a, b] \to \mathbb{R}$ is bounded and $\{P_n : n \in \mathbb{N}\}$ is a sequence of partitions of $[a, b]$ such that $\lim_n(U(P_n; f) - L(P_n; f)) = 0$ then f is integrable. [4 marks]

   (c) Use part 6b above to show that the signum function sgn defined by
   $$\operatorname{sgn}(x) := \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 1 \end{cases}$$
   is Riemann integrable on $[-1, 1]$ and $\int_{-1}^{1} \operatorname{sgn} = 0$. [8 marks]

   (d) Determine whether each of the following statements is true or false. Justify your answers.
   i. If $f : [0, 1] \to \mathbb{R}$ is a bounded function, then f is Riemann integrable. [3 marks]
   ii. There are two distinct functions $f, g : [0, 1] \to \mathbb{R}$ such that the sum $f + g$ is Riemann integrable and yet neither f nor g is Riemann integrable. [3 marks]

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QUESTION 7

7. (a) State and prove the squeeze theorem for sequences of real numbers. [6 marks]

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(x)| \leq 0, \forall x \in \mathbb{R}$. Then, f is a decreasing function. [6 marks]

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ and that $a, b \in \mathbb{R}$ with $a < b$. Let $g, h : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$g(x) := f(b) - f(x) - (b - x)f'(x)$$

and

$$h(x) := (b - a)^2 g(x) - (b - x)^2 g(a)$$

i. Show that $h(a) = h(b)$. [2 marks]

ii. Use Rolle's theorem to show that for some $c \in (a, b)$

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2}(b - a)^2 f''(c)$$

[6 marks]