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UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2010/2011

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER : NUMERICAL ANALYSIS II

COURSE NUMBER : M 411

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :
1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS.
3. NON PROGRAMMABLE CALCULATORS MAY BE USED.

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Use Chebyshev polynomials of the first kind with degree at most 2 to approximate $\arccos(x)$. [10 marks]

   (b) The Chebyshev polynomials $\{U_0(x), U_1(x), \dots\}$ of the second kind are defined by

   $$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)}; \; n = 0, 1, 2, \dots$$

   Show that these functions are orthogonal on the open interval $(-1, 1)$ with respect to the weight function $w(x) = \sqrt{1-x^2}$. [10 marks]

QUESTION 2

2. (a) Let $S := \{\phi_0(x), \phi_1(x), \dots, \phi_n(x)\}$ be a set of orthogonal functions on $[a, b]$ with respect to the weight function $w(x)$. Show that S is linearly independent. [5 marks]

   (b) For the problem of approximating $1/x$ on $[1, 2]$ using a quadratic least squares polynomial, the normal equations may be written in matrix form

   $$A\mathbf{b} = \mathbf{c} \qquad (1)$$

   Explicitly identify matrix A, and vectors $\mathbf{b}$ and $\mathbf{c}$. [10 marks]

   (c) What constant α makes the expression

   $$\sum_{k=0}^{m} [f(x_k) - \alpha x_k + 1]^2$$

   as small as possible? [5 marks]

QUESTION 3

3. (a) Use a single step of the Runge-Kutta method of order 2 to solve:

   $$x'' - x' - 2x = 3e^{-t}, \; 0 \le x \le 1, \; x(0) = 0, x'(0) = 1,$$

   for $x(0.1)$ and $x'(0.1)$. [14 marks]

   (b) Use the Runge-Kutta method of order 4 to find value of the function

   $$x(t) = \int_0^t e^{\tau^2} d\tau$$

   at $t = 0.1$. [6 marks]

QUESTION 4

4. A multi-step method for solving the initial value problem (IVP)

$$y'(x) = f(x,y), \ a \le x \le b, \ y(a) = \alpha$$

is defined by the difference equation

$$y_{n+2} = -3y_n + 4y_{n+1} - 2hf(t_n, y_n); \ n = 0, 1, \ldots, N-2$$

with starting values y_0 and y_1.

(a) Use this method to solve

$$y'(x) = 2 - y, \ 0 \le x \le 0, \ y(0) = 0$$

for $y(0.2)$ and $y(0.3)$ with $h = 0.1$, and starting values $y_0 = 0$ and $y_1 = 2. - e^{-0.1}$. [6 marks]

(b) Write down the local truncation error for the method. [3 marks]

(c) Determine whether or not the method is convergent. [11 marks]

QUESTION 5

5. (a) Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \ 0 \le x \le 2, \ 0 \le y \le 3, \\ u(x,0) &= x/2, \ u(x,3) = 1, \ 0 \le x \le 2, \\ u(0,y) &= y/3, \ u(2,y) = 1, \ 0 \le y \le 3. \end{aligned}$$

Use a finite difference method known as "*the* 5 *point formula*" with a uniform grid on S to approximate both $u(1,1)$ and $u(1,2)$. [10 marks]

(b) Determine a sufficient condition for convergence of the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

for approximating the advection equation

$$u_t + au_x = 0,$$

subject to initial condition $u(x,0) = f(x)$, where $a > 0$ is given.[10 marks]

QUESTION 6

6. Consider the differential problem;

$$u_t(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0,$$
$$u(0,t) = 0,\ u_x(1,t) = u(1,t) - 1, t > 0,$$
$$u(x,0) = \sin(\pi x),\ 0 \le x \le 1.$$

Suppose that an approximate solution to this problem is determined by replacing u_t with a forward difference, and that both u_x and u_{xx} are replaced by central differences.

(a) Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{u}_{j+1} = B\mathbf{u}_j + \mathbf{v}, \text{ where } j = 1, 2, \ldots$$

Identify the square matrix B, and the vectors $\mathbf{u}_j$ and $\mathbf{v}$. [12 marks]

(b) Compute the leading terms of the truncation error for this numerical scheme. [8 marks]

QUESTION 7

7. (a) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{2}\left(\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{(\Delta x)^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{(\Delta x)^2}\right)$$

for approximating the differential equation

$$u_t = u_{xx} \tag{2}$$

is unconditionally stable. [10 marks]

(b) Use the numerical scheme given in part (7a) with $\Delta t = 0.1$ and $\Delta x = 0.5$ to solve parabolic differential equation (2) for $u(0.5, 0.1)$ subject to boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,t) = x(1-x),\ 0 \le x \le 1.$$

[10 marks]