UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2010/2011

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. NON PROGRAMMABLE CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Find a linear least squares polynomial approximation to the function $x\sin(\pi x)$ on the interval $[0, 1]$. [10 marks]

   (b) Given that functions $\{\phi_0(x), \phi_1(x), \phi_2(x)\}$ are orthogonal on $(0, \infty)$ with respect to the weight function $w(x) = 1$, use the Gram-Schmidt process to construct these functions. [10 marks]

QUESTION 2

2. (a) Fit a linear polynomial through the data

i	0	1	2	3
x_i	1.8	2.1	3.2	4.4
y_i	2.0	4.2	8.7	9.5

   in the least squares sense. [10 marks]

   (b) Prove that Chebyshev polynomials $\{T_0(x), T_1(x), \dots\}$ of the first kind are orthogonal on the open interval $(-1, 1)$ with respect to weight function $w(x) = 1/\sqrt{1-x^2}$. [10 marks]

QUESTION 3

3. (a) Use a single step of the modified Euler method to solve:

$$x'' - x' - 2x = 3e^{-t},\ 0 \le x \le 1,\ x(0) = 0, x'(0) = 1$$

   for $x(0.1)$ and $x'(0.1)$. [14 marks]

   (b) Use the Runge-Kutta method of order 2 to find value of the function

$$x(t) = \int_0^t e^{\tau^2} d\tau$$

   at $t = 0.1$. [6 marks]

QUESTION 4

4. A multi-step method for solving the initial value problem (IVP)

$$y'(x) = f(x, y), \ a \le x \le b, \ y(a) = \alpha$$

is defined by the difference equation

$$y_{n+1} = 5y_{n-1} - 4y_n + 2h[f(t_n, y_n) + 2f(t_{n-1}, y_{n-1})]; \ n = 0, 1, \ldots, N-2$$

with starting values y_0 and y_1.

(a) Use this method to solve

$$y'(x) = x + y, \ 0 \le x \le 1, \ y(0) = 0$$

for $y(0.4)$ with $h = 0.1$, and starting values $y_0 = 0$ and $y_1 = 1$. [9 marks]

(b) Determine whether or not the method is convergent. [11 marks]

QUESTION 5

5. (a) Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= x + y, \ 0 \le x \le 2, \ 0 \le y \le 3, \\ u(x, 0) &= u(x, 1) = x, \ 0 \le x \le 2, \\ u(0, y) &= 0, \ u(1, y) = 2, \ 0 \le y \le 3. \end{aligned}$$

Compute finite difference approximations to both $u(1, 1)$ and $u(1, 2)$ by replacing both u_{xx} and u_{yy} with central difference approximations on a uniform grid with step size $h = 1$. [10 marks]

(b) Consider the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

for approximating the advection equation

$$u_t + au_x = 0,$$

subject to initial condition $u(x, 0) = f(x)$, where $a > 0$ is given.
Show that this numerical scheme is convergent provided
$0 \le a\frac{\Delta t}{\Delta x} \le 1.$ [10 marks]

QUESTION 6

6. Consider the differential problem;

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) =& u(0,t) - 1,\ u_x(1,t) = 0, t > 0, \\ u(x,0) =& x(1-x),\ 0 \le x \le 1. \end{aligned}$$

Suppose that an approximate solution to this problem is determined by replacing u_t with a backward difference, and that both u_x and u_{xx} are replaced by central differences.

(a) If resulting finite difference equations may be written in matrix form as

$$\mathbf{u}_j = B\mathbf{u}_{j-1} + \mathbf{v},\ \text{where } j = 1, 2, \ldots$$

then dentify the square matrix B, and the vectors $\mathbf{u}_j$ and $\mathbf{v}$. [12 marks]

(b) Compute the leading terms of the truncation error for this numerical scheme. [8 marks]

QUESTION 7

7. (a) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{(\Delta x)^2}$$

for approximating the differential equation

$$u_t = u_{xx} \tag{1}$$

is stable provided $0 < \dfrac{\Delta t}{(\Delta x)^2} \le \dfrac{1}{2}$. [10 marks]

(b) Use the numerical scheme given in part (7a) with $\Delta t = 0.1$ and $\Delta x = 1$ to solve parabolic differential equation (1) for $u(1, 0.1)$ and $u(2, 0.1)$ subject to boundary conditions

$$u(0,t) = u(3,t) = 0$$

and initial condition

$$u(x,t) = x(x-3),\ 0 \le x \le 3.$$

[10 marks]