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University of Swaziland

Supplementary Examination, July 2011

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions. **Submit solutions to ONLY FIVE questions.**
4. Show all your working.
5. A Table of Laplace Transforms is provided at the end of the question paper.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) By eliminating the arbitrary function find the partial differential equation satisfied by

$$xy + u = f(x^2 + y^2 - u^2).$$

[10 marks]

(b) Find three forms of the general solution for the following partial differential equation

$$u(u^2 + xy)(xu_x - yu_y) = y^4.$$

[10 marks]

Question 2

Consider the partial differential equation

$$25u_{xx} + 20u_{xy} + 4u_{yy} = 24x$$

(a) Classify the partial differential equation as hyperbolic, parabolic or elliptic. [5 marks]

(b) Reduce the equation into its canonical form and hence find its general solution. [15 marks]

Question 3

(a) Find the particular solution of the partial differential equation

$$yu_x - x^2u_y = xy$$

which contains the curve $u = x^2$ on $3y^2 = 2x^3$. [8 marks]

(b) Show that

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial F}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial F}{\partial \varphi}\right)^2$$

under the transformation $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. [12 marks]

Question 4

Consider the function

$$f(x) = \begin{cases} -1, & -\pi \le x < 0; \\ 0, & x = 0; \\ +1, & 0 < x \le \pi. \end{cases}$$

(a) Find the fourier series expansion for $f(x)$. [10 marks]

(b) Use Parseval's identity to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

[10 marks]

Question 5

Solve the boundary value problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0, && 0 < x, y < 1 \\ &u(0, y) = y, && 0 \le y \le 1 \\ &u(1, y) = 0, && 0 \le y \le 1 \\ &u(x, 0) = 0, && 0 \le x \le 1 \\ &u(x, 1) = 0, && 0 \le x \le 1 \end{aligned}$$

[20 marks]

Question 6

Solve the following equations using the method of Laplace transforms

(a)

$$\begin{aligned} &u_{xt} + \sin t = 0, \quad x > 0, \quad t > 0 \\ &u(x, 0) = x, \quad x \ge 0 \\ &u(0, t) = 0, \quad t \ge 0 \end{aligned}$$

[10 marks]

(b)

$$\begin{aligned} &x u_x + u_t = xt, \quad x > 0, \quad t > 0 \\ &u(x, 0) = 0, \quad x \ge 0 \\ &u(0, t) = 0, \quad t \ge 0 \end{aligned}$$

[10 marks]

Question 7

Solve the Dirichlet problem inside the circle

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \quad -\pi < \theta < \pi$$

$$u(1,\theta) = \begin{cases} 1, & -\pi < x < 0; \\ \frac{1}{2}, & 0 < x < \pi. \end{cases}$$

[20 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$