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# University of Swaziland

## Final Examination, 2010/2011

## BSc IV, Bass IV, BEd IV

**Title of Paper** : Abstract Algebra II

**Course Number** : M423

**Time Allowed** : Three (3) hours

**Instructions** :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

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## QUESTION 1

(a) Define a unit in a ring $R$. [2]

(b) Find a polynomial $f(x) \in \mathbb{Z}_4[x]$ with $\deg[(f(x)] > 0$ that is a unit. [5]

(c) Prove that if $R$ is a ring with unity and $I$ an ideal in $R$ containing a unit, then $I = R$. [7]

(d) Write $x^3 + 3x^2 + 3x + 4 \in \mathbb{Z}_5[x]$ as a product of irreducible polynomials. [6]

## QUESTION 2

(a) Show that the map $\phi : \mathbb{C} \to M_2(\mathbb{R})$ defined by

$$\phi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is a ring homomorphism. Find its kernel. [6]

(b) Find all the units in

i. $\mathbb{Z}[x]$ ii. $\mathbb{Z}_7[x]$ [4]

(c) Show that the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a divisor of zero in $M_2(\mathbb{Z})$. [5]

(d) Show that the set $T = \{2a + 1 : a \in \mathbb{Z}\}$ together with addition and multiplication as defined on $\mathbb{Z}$ does not form a ring. [5]

## QUESTION 3

(a) Consider the Evaluation Homomorphism $\phi_\alpha : F[x] \to E$, where $F$ is a subfield of the field $E$. Let $F = \mathbb{Q}$ and $E = \mathbb{R}$. Verify that

$$x^2 + x - 6 \in \text{Ker}(\phi_2).$$

[2]

(b) Which of the following polynomials in $\mathbb{Z}[x]$ satisfy Eisensten's criterion for irreducibility over $\mathbb{Q}$?

i. $4x^{10} - 9x^3 + 24x - 18$

ii. $2x^{10} + 25x^3 + 10x^2 - 30$ [6]

(c) Deduce the irreducibility or otherwise of

$$x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x].$$

[6]

(d) The polynomial $x^2 + x + 1$ has a zero $\alpha$ in $\mathbb{Z}_2(\alpha)$ and therefore must factor into linear factors in $(\mathbb{Z}_2(\alpha))[x]$. Find this factorisation. [6]

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## QUESTION 4

(a) Let $R$ be a ring and $S$ a subset of $R$. Show that $S$ is a subring of $R$ if and only if

i. $S$ is closed with respect to the operations of $R$, and
ii. for each $a \in S$, we have $-a \in S$. [10]

(b) Determine whether the indicated operations of addition and multiplication are closed on the set. If they are, does the set together with the operations form a ring?

i. $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ with addition and multiplication as defined on $\mathbb{Z}$.
ii. $\mathscr{A} = \{A \in M_2(\mathbb{R}) : \det(A) = 0\}$ with the usual addition and multiplication of matrices. [10]

## QUESTION 5

(a) Prove that every finite integral domain is a field. [10]

(b) Show that for a field $F$, the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $a, b \in F$ is a right ideal but not a left ideal in $M_2(F)$. [10]

## QUESTION 6

(a) State (but do not prove) Kronecker's Theorem. [4]

(b) For each of the given algebraic number $\alpha \in \mathbb{C}$, find $\operatorname{irr}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$.

i. $\sqrt{\frac{1}{3} + \sqrt{7}}$
ii. $\sqrt{2} + i$ [8]

(c) Let $R$ be a ring with unity $1 \neq 0$. Prove:

i. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$, then $R$ has characteristic 0. [3]
ii. If $n \cdot 1 = 0$ for some $n \in \mathbb{Z}^+$, then the smallest such integer is the characteristic of $R$. [5]

## QUESTION 7

(a) Let $R$ be a ring with additive identity 0. Prove that for $a, b \in R$,

i. $a0 = 0a = 0$. [4]
ii. $a(-b) = (-a)b = -ab$. [4]
iii. $(-a)(-b) = ab$. [4]

(b) Show that $x^4 - 2x^2 + 8x + 1$ is irreducible over $\mathbb{Q}$. [8]

END OF EXAMINATION PAPER