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University of Swaziland

Supplementary Examination, July 2011

BSc IV, Bass IV, BEd IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

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QUESTION 1

(a) What is a divisor of zero in a ring R? [2]

(b) Find $q(x)$ and $r(x)$ as described by the division algorithm so that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$ for

$$f(x) = x^5 - 2x^4 + 3x - 5, \quad g(x) = 2x + 1 \quad \text{in } Z_{11}[x].$$

[5]

(c) Let R be a commutative ring. For an arbitrary element p in R, form the set $P = \{pr : r \in R\}$. It is clear that $0 \in P$ (take $r = 0$ in R to get $p0 = 0$ in P).

i. Prove that P is a subring of R by showing that P is closed with respect to the ring operations and that each element of P has its additive inverse in P. [5]

ii. Show that P is an ideal in R. [4]

(d) Find all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[x]/\langle x^3 + cx^2 + 1\rangle$ is a field. [4]

QUESTION 2

(a) Determine the irreducibility or otherwise of

i. $x^3 + 3x^2 - 8$ in $\mathbb{Q}[x]$ [6]

ii. $8x^3 + 6x^2 - 9x + 24$ in $\mathbb{Q}[x]$. [4]

(b) For $a, b \in \mathbb{Z}$, define $a \oplus b = a + b + 1$ and $a \odot b = a + b + ab$. Show that $\mathbb{Z}$ is a commutative ring with respect to $\oplus$ and $\odot$. [10]

QUESTION 3

(a) Prove that every finite integral domain is a field. [10]

(b) Find all the units in

i. $\mathbb{Z}_4$ ii. $\mathbb{Z}_7[x]$ [2,3]

(c) Show that the set $S = \{2a : a \in \mathbb{Z}\}$ with addition and multiplication as defined on $\mathbb{Z}$ is a ring. [***Hint:*** First show that the set is closed with respect to the ring operations.] [5]

QUESTION 4

(a) In the ring $\mathbb{Z}_n$ prove that

i. the divisors of zero are those nonzero elements that are NOT relatively prime to n, (5)

ii. the elements that are relatively prime to n cannot be divisors of zero. (5)

(b) Determine whether the indicated operations of addition and multiplication are closed on the set. If they are, does the set together with the operations form a ring?

i. $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ with addition and multiplication as defined on $\mathbb{Z}$.

ii. $\mathscr{A} = \{A \in M_2(\mathbb{R}) : \det(A) = 0\}$ with the usual addition and multiplication of matrices. [10]

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QUESTION 5

(a) Define

i. the *characteristic* of a ring, [3]

ii. a *ring homomorphism*, [2]

iii. a *unit* in a ring with unity. [2]

(b) Consider the Evaluation Homomorphism $\phi_\alpha : F[x] \to E$, where F is a subfield of the field E. Let $F = \mathbb{Q}$ and $E = \mathbb{C}$. Show that

$$x^2 + 1 \in \mathrm{Ker}(\phi_i).$$

[2]

(c) Prove that every field is an integral domain. [7]

(d) Does the polynomial $4x^{10} - 9x^3 + 24x - 18$ satisfy Eisensten's criterion for irreducibility over $\mathbb{Q}$? Explain. [4]

QUESTION 6

(a) In each case, give an example of a ring satisfying the given condition.

i. A commutative ring with zero divisors.

ii. A ring with no zero divisors.

iii. A ring that is not a division ring.

[6]

(b) For each of the given algebraic number $\alpha \in \mathbb{C}$, find $\mathrm{irred}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$.

i. $\sqrt{\frac{1}{3} + \sqrt{7}}$

ii. $\sqrt{2} + i$

[8]

(c) Let $\phi : R \to R'$ be ring homomorphism. Show that ϕ is one-to-one and onto if and only if $\ker(\phi) = \{0\}$ [6]

QUESTION 7

(a) Is $\mathbb{Q}[x]/\langle x^2 - 6x + 6\rangle$ a field? Explain. [5]

(b) i. Show that $x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$. [2]

ii. Let α be a zero of $x^2 + x + 1$ in the extension field of $\mathbb{Z}_2(\alpha)$ of $\mathbb{Z}_2$. List the elements of $\mathbb{Z}_2(\alpha)$ and give the addition and multiplication tables of $\mathbb{Z}_2(\alpha)$. [10]

iii. Find the factorisation of $x^2 + x + 1$ into linear factors in $\mathbb{Z}_2(\alpha)$. [3]

END OF EXAMINATION PAPER