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UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2010/11

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set with a map $d : X \times X \longrightarrow \mathbb{R}$. What is meant by saying that (X, d) is a *metric space*?

(b) Check carefully that the New York distance is a metric on $\mathbb{R}^2$. [20]

QUESTION 2

(a) Let $x(t) = t$ and $y(t) = t^2$ be continuous functions on $\mathbb{R}$. Find $d(x, y)$ where d is:

(i) the uniform metric on $C[0, 1]$;

(ii) the L_1–metric on $C[0, 2]$;

(iii) the L_2–metric on $C[0, 2]$. [10]

(b) Define what is meant by:

(i) a *Cauchy sequence* in a metric space,

(ii) a *complete metric space.* [4]

(c) Which of the following spaces X is complete and which is incomplete in the usual (Euclidean) metric? Give reasons.

(i) $X = \mathbb{Q}$,

(ii) $X = \{\frac{1}{n} : n \in \mathbb{N}\}$. [6]

QUESTION 3

Let $x = (3, 4)$, $y = (-2, 9)$, and $z = (4.5, 6)$ in $\mathbb{R}^2$. Find $d(x, y)$, $d(x, z)$ and $d(y, z)$ in each of the following metrics on $\mathbb{R}^2$:

(a) Euclidean metric;

(b) Max metric;

(c) London (or UK rail) metric;

(d) Chicago metric;

(e) New York metric;

(f) Raspberry pickers (or lift) metric. [20]

QUESTION 4

(a) Let (X, d) be a metric space and let $S \subseteq X$. What is meant by saying that S is closed? Prove that any intersection of closed sets in X is closed and any finite union of closed sets in X is closed. [8]

(b) What is meant by an *open ball* $B(a, r)$ in a metric space (X, d)? Show that an open ball is open. By drawing a diagram, or otherwise, describe the open ball $B(a, 3)$ in $\mathbb{R}^2$, where $a = (4, 5)$

(i) with the usual metric

(ii) with the max metric. [6]

(c) Prove that in any metric space X, each closed ball is a closed set. Show that any finite set in X is closed. [6]

QUESTION 5

(a) Let (X,d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\dfrac{n^2}{2n^2+1}, \dfrac{1}{n+1}\sin(\dfrac{n\pi}{2})\right)$,

(ii) $x_n = (3^{-n}, (-1)^n \exp(\dfrac{1}{n}))$. [8]

(c) (i) Suppose that (x_n) converges to x in $C[a,b]$ in the uniform metric. Explain what is meant by *pointwise convergence*. Show that (x_n) converges to x pointwise.

(ii) Let x_n in $C[0,1]$ be defined by

$$x_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \le t \le 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } 0 < t \le 1. \end{cases}$$

Deduce that (x_n) is not convergent in $C[0,1]$. [10]

QUESTION 6

(a) Let X be a metric space and $A \subseteq X$. What is meant by saying that A is *compact*? [2]

(b) Assuming that a closed bounded subset of $\mathbb{R}$ is compact, show that the same is true for $\mathbb{R}^2$. [8]

(c) Show that in any metric space, a closed subset of a compact set is compact. [4]

(d) Which of the following sets is compact? Give reasons.

(i) $\{(x, y) : 0 \leq x < y \leq 1\}$ in $\mathbb{R}^2$,

(ii) $\{1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\}$ in $\mathbb{R}$. [6]

QUESTION 7

(a) Prove that in a metric space (X, d), a subset $F \subseteq X$ is closed if the limit of any convergent sequence (x_n) of points of F is in F. [8]

(b) Prove that $\mathbb{R}^2$ equipped with the metric

$$d(x, y) = \alpha|x_1 - y_1| + |x_2 - y_2|, \qquad x = (x_1, x_2), \quad y = (y_1, y_2)$$

is complete, where $\alpha > 0$ is fixed. [12]

END OF EXAMINATION