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UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2010/11

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set with a map $d : X \times X \longrightarrow \mathbb{R}$. What is meant by saying that (X, d) is a *metric space*?

(b) Check carefully that the Raspberry pickers' distance is a metric on $\mathbb{R}^2$. [20]

QUESTION 2

(a) Let (X, d) be a metric space and let $S \subseteq X$. What is meant by saying that S is open? Prove that any union of open sets in X is open and any finite intersection of open sets in X is open. [8]

(b) What is meant by an *open ball* $B(a, r)$ in a metric space (X, d)? Show that an open ball is open. By drawing a diagram, or otherwise, describe the open ball $B(a, 3)$ in $\mathbb{R}^2$, where $a = (2, 5)$

(i) with the usual metric,

(ii) with the max metric. [6]

(c) Show that $\emptyset$ and X are closed, where (X, d) is a metric space. [6]

QUESTION 3

Let $A = \{(x_1, x_2) : 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 2\}$ and let $x = (2, 2)$. Find $d(x, A)$ for each of the following metrics:

(a) Euclidean metric;

(b) Max metric;

(c) London (or UK rail) metric;

(d) Chicago metric;

(e) New York metric;

(f) Raspberry pickers (or lift) metric.

Calculate diam(A) in each case. [20]

QUESTION 4

(a) Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. What is meant by saying that $\{x_n\}$ is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\frac{n^3}{3n^3+1}, \frac{3}{3n+3}\sin(\frac{n\pi}{2})\right)$,

(ii) $x_n = (10^{-n}, (-1)^n \exp(\frac{1}{n}))$. [8]

(c) (i) Suppose that $\{x_n\}$ converges to x in $C[a, b]$ in the uniform metric. Explain what is meant by *pointwise convergence* of a sequence $\{x_n\}$ in $C[a, b]$. Show that $\{x_n\}$ converges to x pointwise.

(ii) Let x_n in $C[0, 1]$ be defined by

$$x_n(t) = \begin{cases} \frac{nt}{n-1} & \text{if } 0 \leq t \leq 1 - \frac{1}{n}, \\ n(1-t) & \text{if } 1 - \frac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that $\{x_n\}$ converges pointwise to the function

$$x(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Deduce that $\{x_n\}$ is not convergent in $C[0, 1]$ in the uniform metric. [10]

QUESTION 5

(a) Find $d(x,y)$, where d is the

(i) uniform metric,

(ii) the L_1−metric

on $C[-1,1]$, $x(t)=t$ and $y(t)=t^2$. [8]

(b) Suppose that $f,g : X \longrightarrow \mathbb{R}$ are both continuous. Show that the function $h : X \longrightarrow \mathbb{R}$ defined by

$$h(x) = 6f(x) - 5g(x)$$

is continuous. [4]

(c) Let f be the function $f : C[0,1] \longrightarrow \mathbb{R}$ defined for $x \in C[0,1]$ by $f(x) = x(0)$. Show that f is not continuous with respect to the L_1 metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions $x_n(t)$ given by

$$x_n(t) = \begin{cases} (n-1)t & \text{if } 0 \leq t \leq \frac{1}{n} \\ 1-t & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

(**Hint** Sketch the functions $x_n(t)$ and consider their limit in the L_1 metric). [8]

QUESTION 6

(a) (i) Explain what is meant by a contraction of a metric space, and state without proof the Contraction Mapping Theorem.

(ii) Show that the mapping $f : [-1,1] \longrightarrow [-1,1]$ defined by $f(x) = \frac{1}{14}(x^4 - 3x^3 + 9)$ is a contraction, and deduce that there is unique solution to the equation $x^4 - 3x^3 - 14x + 9 = 0$ in the interval $[-1,1]$. [10]

(b) Is the set $X = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ complete or incomplete? Justify your answer. [2]

(c) Explain what is meant by saying that a metric space X is *connected*. Which of the following subspaces of $\mathbb{R}$ is connected and which is disconnected? Give reasons. (Any theorem about connected subsets of $\mathbb{R}$ that you use should be stated carefully but not proved)

(i) $\mathbb{R} - \mathbb{Q}$,

(ii) $(2,5) \cup (3,\infty)$,

(iii) $[99,101)$. [8]

QUESTION 7

(a) Prove that in a metric space (X, d), if a subset $F \subseteq X$ is closed, then the limit of any convergent sequence $\{x_n\}$ of points of F is in F. [6]

(b) Consider $\mathbb{R}^2$ with the New York metric, and let $(x^{(n)})_{n\geq 1}$ be a sequence of points in $\mathbb{R}^2$; where, for each $j \in \mathbb{N}$, $x^{(j)} = (x_1^{(j)}, x_2^{(j)})$. Let $x = (x_1, x_2)$ be a point in $\mathbb{R}^2$. Show that $(x^{(n)})_{n\geq 1}$ converges to x if and only if: either $x_1^{(n)} = x_1 \, \forall n \in \mathbb{N}$ and $x_2^{(n)} \to x_2$ as $n \to \infty$; or $x_1^{(n)} \neq x_1$ and $x_1^{(n)} \to x_1$, $x_2^{(n)} \to x_2$ as $n \to \infty$, while $x_2 = 0$. [9]

(c) Consider the sequence $x_n = \left(3 - \frac{1}{2^n}, \frac{1}{n}\right)$ in $\mathbb{R}^2$. Decide whether $(x_n)_{n\geq 1}$ converges to $(3, 0)$ with:

(i) the max metric;

(ii) the New York metric. [3,2]

END OF EXAMINATION