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University of Swaziland

Final Examination, 2011/12

BSc II, Bass II, BEd II, BEng

Title of Paper : Calculus II

Course Number : M212

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Express the given rectangular equations in polar

(i) $xy = 4$

(ii) $x^2 - 8x + y^2 + 7 = 0$

[4]

(b) Consider the curve

$$r = 2 + 2\sin\theta.$$

(i) Sketch the curve.

(ii) Find the area enclosed by the curve.

(iii) Find the length of the curve.

[16]

Question 2

(a) Find the equation of the tangent to the surface

$$f(x, y) = x^2 + 3y^2 - 4z^2 + 3xy - 10yz + 4x - 5z - 22$$

at the point $(1, -2, 1)$. [10]

(b) Find and classify the critical points of

$$f(x, y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 2.$$ [10]

Question 3

(a) Consider Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

where $z = f(x, y)$. Show that under the transformation $x = r\cos\theta$, $y = r\sin\theta$, Laplace's equation takes the form

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0. \qquad [10]$$

(b) Find the point on the plane

$$x + 2y - 3x - 4 = 0$$

which is nearest to the origin. [10]

Question 4

(a) Show that each of the specified functions satisfies the given partial differential equation.

(i) $f(x, y) = \sqrt{x^2 + y^2}$ satisfies $xf_x + yf_y = 0$ [5]

(ii) $f(x, y) = e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} \cos\left(\frac{y}{x}\right)$ satisfies

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0. \qquad [5]$$

(b) Evaluate

$$\int\int_R xy^2 \mathrm{d}x\mathrm{d}y$$

where R is bounded by $x + y + 1 = 0$ and $x + y^2 = 1$. [10]

Question 5

(a) For each of the following use a double integral to find the area bounded by the curves.

(i) $y = x^3 + 8, \quad y = 4x + 8$

(ii) $x = y^2 - 2y, \quad y + x = 12$

[10]

(b) Find the directional derivatives of

(i) $f(x, y, z) = x^2 + y^2 + z^2$ at the point $P_0(x_0, y_0, z_0) = P_0(3, 2, 1)$ in the direction $\vec{v}$ from the point $(1, 0, 1)$ to $(2, -1, 3)$. [5]

(ii) $f(x, y, z) = x^2 y + xz$ at $P_0(-1, 1, -1)$ in the direction of the vector from $(3, 2, 1)$ to $(3, 1, -1)$. [5]

Question 6

(a) Show that the ellipsoid $3x^2 + 3y^2 + 8z^2 - 34 = 0$ and the hyperboloid of two piece $4x^2 - 4y^2 - z^2 - 4 = 0$ are orthogonal (perpendicular) to each other at the common point $\left(\frac{4}{5}\sqrt{5}, \sqrt{2}, \frac{2}{5}\sqrt{5}\right)$. [10]

(b) Show that the function

$$f(x, y) = \frac{xy}{x - y}$$

satisfies

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 0. \qquad [10]$$

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Question 7

Evaluate

(a) $$\int_0^1 \int_0^{x^2} \int_{xy}^{x+y} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$ [10]

(b) $$\int_0^8 \int_0^{\sqrt{81-y^2}} \int_0^{\sqrt{81-y^2-x^2}} \frac{1}{\sqrt{x^2+y^2}}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$ [10]