UNIVERSITY OF SWAZILAND

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FINAL EXAMINATION 2011/2012

B.A.S.S. /BEd. /BEng. /BSc. II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

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QUESTION 1

1. (a) i. Compute the inverse A^{-1} of the square matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix}$. [10 marks]

ii. Use A^{-1} obtained in 1(a)i to solve the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 9 \\ x & + & 3y & + & 5z & = & -11 \\ x & + & 3y & + & 4z & = & 13 \end{array}$$

[3 marks]

iii. Can A^{-1} from 1(a)i be expressed as a product of elementary matrices? Justify your answer. [2 marks]

(b) Determine the value of k for which the matrix

$$\begin{bmatrix} 1-k & -1 & 2 \\ 0 & 4-k & 3 \\ 0 & 4 & -k \end{bmatrix}$$

is invertible. [5 marks]

QUESTION 2

2. (a) Consider the linear system

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & cx_2 & + & 6x_3 & = & 6 \\ -x_1 & + & 3x_2 & + & (c-3)x_3 & = & 0 \end{array} \qquad (1)$$

i. Find values of c for which the linear system (1) has;

A. no solutions,

B. a unique solution,

C. infinitely many solutions. [10 marks]

ii. In the case of 2(a)iC write down the general solution. [3 marks]

(b) Given that x and y are real numbers with $x < y$, uniquely determine the matrix $A = \begin{bmatrix} x & 1 \\ 2 & y \end{bmatrix}$ which satisfies the equation $A^2 + 5A = -4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. [7 marks]

QUESTION 3

3. (a) What does it mean to say that a non-empty set U is a subspace of a vector space V? [4 marks]

   (b) Determine whether or not the following subsets are subspaces. Justify your answers.

      i. $U = \{(x, y, z) \in \mathbb{R}^3 : x + yz = 0\}$ in $\mathbb{R}^3$. [4 marks]

      ii. $U = \{p(x) \in P_2 | p(0) = p'(0) = 0\}$ in the set P_2 of all polynomials of degree at most 2. [4 marks]

      iii. $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b + c = 0 \right\}$ in the set M_{22} of all of all 2×2 matrices. [4 marks]

      iv. $U = \{(a, b + c, b - c) \in \mathbb{R}^3 : a, b, c \in \mathbb{R}\}$ in $\mathbb{R}^3$. [4 marks]

QUESTION 4

4. (a) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a set of vectors in a vector space V.

      i. Explain the statement "S spans V". [2 marks]

      ii. What does it mean to say that S is linearly independent in V? [2 marks]

      iii. What does it mean to say that S is a basis for V. [2 marks]

   (b) Consider the vectors $\mathbf{u}_1 = (1, 2, 4), \mathbf{u}_2 = (1, 1, 1), \mathbf{u}_3 = (1, -1, 3)$ in $\mathbb{R}^3$.

      i. Determine whether or not $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbb{R}^3$. [4 marks]

      ii. Determine whether or not the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are linearly independent in $\mathbb{R}^3$. [2 marks]

      iii. Does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ form a basis for $\mathbb{R}^3$? Justify your answer. [2 marks]

   (c) i. Define the column space of an $m \times n$ matrix. [2 marks]

      ii. Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ -1 & -2 & 1 & 0 & -2 \\ 1 & 2 & 1 & 2 & 4 \end{bmatrix}$$

[4 marks]

QUESTION 5

5. (a) i. Let U and V be vector spaces, and let $T : U \to V$ be a linear transformation.

A. Define the image of T. [2 marks]

B. Define the kernel of T. [2 marks]

C. Define the rank and nullity of T, and state carefully a theorem that relates the two. [4 marks]

ii. Let M_{22} denote the space of all real-valued 2×2 matrices, and let $T : M_{22} \to M_{22}$ be a map defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+c & 0 \\ b+c+d & c+d \end{bmatrix}$$

A. Show that T is a linear map. [4 marks]

B. Find bases for the image and kernel of T. [6 marks]

C. Verify the theorem mentioned in 5(a)iC for T. [2 marks]

QUESTION 6

6. (a) Let P_2 denote the set of all polynomials of degree at most 2. If $p(x), q(x) \in P_2$ then write $p(x) = p_2x^2 + p_1x + p_0$ and $q(x) = q_2x^2 + q_1x + q_0$. Determine whether or not the following are inner products on P_2. Justify your answers.

i.

$$\langle p, q \rangle = \int_0^1 (1-x)p(x)q(x)dx$$

[6 marks]

ii.

$$\langle p, q \rangle = p_2q_2 + p_0q_0$$

[6 marks]

(b) Consider the inner product space consisting of the vector space M_{22} (of all real-valued 2×2 matrices) together with an inner product defined by

$$\langle A, B \rangle = \mathrm{Tr}(B^T A)$$

i. Define the norm $\|A\|$ of a matrix $A \in M_{22}$ with respect to this inner product. [2 marks]

ii. Compute $\|A\|$ when $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. [2 marks]

iii. Verify the Cauchy-Schwarz inequality for the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

in this inner product space. [4 marks]

QUESTION 7

7. (a) Given an $n \times n$ matrix A, what is meant by "an eigenvector and an eigenvalue of A"? [4 marks]

(b) For the matrix

$$A = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$$

find its eigenvalues and corresponding eigenvectors. [10 marks]

(c) i. State the Cayley-Hamilton theorem. [2 marks]

ii. Verify the Cayley-Hamilton theorem with $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. [4 marks]