UNIVERSITY OF SWAZILAND

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SUPPLEMENTARY EXAMINATION 2011/2012

B.A.S.S. /BEd. /BEng. /BSc. II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

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QUESTION 1

1. (a) Use Gaussian elimination to solve the linear system

$$\begin{array}{rcrcrcr} x & + & 3y & + & 4z & = & 13 \\ x & + & 2y & + & 3z & = & 9 \\ x & + & 3y & + & 5z & = & -11 \end{array}$$

[8 marks]

(b) Express the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

as a product of elementary matrices. [12 marks]

QUESTION 2

2. (a) Consider the linear system

$$\begin{array}{rcrcrcr} -x_1 & + & 3x_2 & + & 2x_3 & = & -8 \\ 3x_1 & + & 3x_2 & + & \alpha x_3 & = & \beta \\ x_1 & & & + & x_3 & = & 2 \end{array} \qquad (1)$$

i. Find values of α and β for which the linear system (1) has;

A. no solutions,

B. a unique solution,

C. infinitely many solutions. [10 marks]

(b) Find all real numbers x and y such that

$$\begin{vmatrix} 0 & y & x \\ x & 0 & y \\ x & y & 0 \end{vmatrix} = 0$$

[5 marks]

(c) Let $A = (a_{ij})$ be a 4×4 matrix with

$$a_{ij} = \begin{cases} 0 & \text{if} \quad i > j, \\ j - i + 1 & \text{if} i \leq j. \end{cases}$$

Write down A explicitly.

Is A invertible? Justify your answer. [5 marks]

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QUESTION 3

3. (a) Write down the **subspace test** for determining whether or not a non-empty set W is a subspace of a vector space V. [4 marks]

   (b) Determine whether or not the following subsets are subspaces. Justify your answers.

   i. $U = \{(x, 0, y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$. [4 marks]

   ii. $U = \{p(x) \in P_1 : p' \equiv 0\}$ in the set P_1 of all polynomials of degree at most 1. [4 marks]

   iii. $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 0 \right\}$ in the set M_{22} of all of all 2×2 matrices. [4 marks]

   iv. $U = \{(a, b, c, d) \in \mathbb{R}^4 : a - b + c - d = 0\}$ in $\mathbb{R}^4$. [4 marks]

QUESTION 4

4. (a) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a set of vectors in a vector space V.

   i. Explain the statement "S spans V". [2 marks]

   ii. What does it mean to say that S is linearly independent in V? [2 marks]

   iii. What does it mean to say that S is a basis for V. [2 marks]

   (b) Consider the set $S := \{(1, 2, 0), (1, 2, 3), (6, 5, 4)\}$ of vectors in $\mathbb{R}^3$.

   i. Determine whether or not S spans $\mathbb{R}^3$. [4 marks]

   ii. Determine whether or not S is linearly independent in $\mathbb{R}^3$. [2 marks]

   iii. Is S a basis for $\mathbb{R}^3$? Justify your answer. [2 marks]

   (c) i. Define the row space of an $m \times n$ matrix. [2 marks]

   ii. Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 2 & 4 \\ -1 & -2 & 1 & 0 & -2 \end{bmatrix}$$

[4 marks]

QUESTION 5

5. (a) i. Let U and V be vector spaces, and let $T : U \to V$ be a linear transformation.
   A. Define the image of T. [2 marks]
   B. Define the kernel of T. [2 marks]
   C. Define the rank and nullity of T, and state carefully a theorem that relates the two. [4 marks]

   ii. Are the following linear maps? Justify your answers.
   A. $T : \mathbb{R}^2 \to P_1$ with $T(a, b) = ax + b$. [4 marks]
   B. $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (x + y, xy)$. [4 marks]
   C. $T : P_2 \to P_1$ with $T(ax^2 + bx + c) = 2ax + b$. [4 marks]

QUESTION 6

6. (a) Determine whether or not the following are inner products on the given vector spaces. Justify your answers.

   i.
$$\langle A, B\rangle = \mathrm{Tr}(B^T A)$$
   on the set M_{22} of all 2×2 matrices with standard addition and scalar multiplication for matrices. [6 marks]

   ii.
$$\langle p, q\rangle = \int_0^1 xp(x)q(x)dx$$
   on the set P_2 of all polynomials of degree at most 2 with standard addition and scalar multiplication. [6 marks]

   (b) Consider the inner product space consisting of the vector space $\mathbb{R}^2$ together with an inner product defined by
$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{v}^T \mathbf{u}$$

   i. Define the norm $\|\mathbf{u}\|$ of a vector $\mathbf{u} \in \mathbb{R}^2$ with respect to this inner product. [2 marks]

   ii. Compute $\|\mathbf{u}\|$ when $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. [2 marks]

   iii. Verify the Cauchy-Schwarz inequality for the vectors
$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
   in this inner product space. [4 marks]

QUESTION 7

7. (a) Given an $n \times n$ matrix A, what is meant by "an eigenvector and an eigenvalue of A"? [4 marks]

(b) For the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

find its eigenvalues and corresponding eigenvectors. [10 marks]

(c) Let λ be an eigenvector of a square matrix A with corresponding eigenvector $\mathbf{x}$.

i. Show that $\mathbf{x}$ is also an eigenvector of A^{-1} and the eigenvalue is $\frac{1}{\lambda}$ provided $\lambda \neq 0$. [3 marks]

ii. Also show that $\mathbf{x}$ is an eigenvector of A^2 and the eigenvalue is λ^2. [3 marks]