UNIVERSITY OF SWAZILAND

60

FINAL EXAMINATIONS 2011/2012

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Using the axioms given below, prove each of the theorems which follow.

Axiom 1. All students work hard.

Axiom 2. People who work hard are successful.

Axiom 3. Successful people are not foolish.

Axiom 4. Unhappy people are foolish.

Theorem 1 *If Bill works hard, then he will be happy.*

Theorem 2 *If Bill is unhappy, then he is not a student.*

[4]

(b) (i) State the *Pigeonhole Principle.* [2]

(ii) Prove that at a party of $n \geq 2$ people, there are at least two people who have the same number of friends at the party (where the relation of being friends is assumed not to be reflexive). [6]

(c) Prove by the *direct uniqueness* method that if $x > 2$ is a real number, then there is a unique real number $y < 0$ such that $x = \dfrac{2y}{1+y}$. [4]

(d) Prove by the indirect uniqueness method that if m and b are real numbers such that $m \neq 0$, then there is a unique real number z such that $mz + b = 0$. [4].

QUESTION 2

(a) What do you understand by the following?

(i) Argument; [1]

(ii) Valid argument; [1]

(iii) Invalid argument; [1]

(iv) Mathematical Proof. [1]

(b) State the difference between a proof by contradiction and a proof by the contrapositive method. [2]

(c) Let a be an integer. Prove that if a^2 is divisible by 3, so is a. [6]

(d) Prove that $\sqrt{3}$ is irrational. [8]

QUESTION 3

(a) Let x be an irrational number. Prove that there exists a monotone decreasing sequence $(x_n)_{n\geq 0}$ of rational numbers which converges to x. [10]

(b) Four intelligent frogs sit on a log; two green frogs on one side and two brown frogs on the other side, with an empty seat separating them. They decide to switch places. The only moves permitted are to jump over one frog of a different color into an empty space or to jump into an adjacent space. What is the minimum number of moves? Generalize this problem and solve it. [10]

63

QUESTION 4

(a) Give the definition of a *tautology*. [1]

(b) Show that the proposition

$$[(P \implies Q) \wedge (Q \implies R)] \implies [P \implies R],$$

where P, Q and R are statements, is a tautology. [9]

(c) Let P be the statement "All girls are good at mathematics." Which of the following statements is the negation of P?

(i) All girls are bad at mathematics;

(ii) All girls are not good at mathematics;

(iii) Some girl is bad in mathematics;

(iv) Some girl is not good at mathematics;

(v) All children who are good at mathematics are girls;

(vi) All children who are not good at mathematics are boys;

Can you find any statement in this list that has the same meaning as statement P? [4]

(d) Write, symbolically, the negation of the statement

$$\forall\, x \in X\; \exists\, n_0 \in \mathbb{N}, \forall\, n > n_0\; \exists\, \varepsilon > 0, |f_n(x) - f(x)| < \varepsilon.$$

[6]

64

QUESTION 5

(a) (i) Define the composition $f \circ g$ of any two functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = \cos x$ and $g(x) = x^3 - 1$ for all $x \in \mathbb{R}$. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$. [5]

(b) Let S, T, U be sets and let $f : S \longrightarrow T$ and $g : T \longrightarrow U$ be functions. Prove that if f and g are both bijections, then so is $g \circ f$. [7]

(c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 4 - x$ for all $x \in \mathbb{R}$. Show that f is a bijection and find f^{-1}. [6]

QUESTION 6

(a) Define an *equivalence relation* [3]

(b) Which of the following are equivalence relations on the given set S? [17]

(i) $S = \mathbb{R}$, and $a \sim b \iff a = b$ or $a = -b$.

(ii) $S = \mathbb{Z}$, and $a \sim b \iff ab = 0$.

(iii) S is the set of all points in the plane, and $a \sim b$ means a and b are the same distance from the origin.

(iv) $S = \{1, 2, 3\}$, and $a \sim b \iff a = 1$ or $b = 1$.

(v) $S = \mathbb{R} \times \mathbb{R}$, and $(x, y) \sim (a, b) \iff x^2 + y^2 = a^2 + y^2$.

65

QUESTION 7

(a) Let X and Y be any sets, and let $f : X \longrightarrow Y$ be any mapping from X to Y. Prove that:

(i) $f(A \cup B) = f(A) \cup f(A)$; [5]

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$. [5]

(b) Let A, B and C be any sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Using this and the laws of set intersection and set union, or otherwise, prove that if A and B are subsets of a set X with $A \cap B \neq 0$, then

$$(A^c \cup B) \cap (A \cup B^c) = (A^c \cap B^c) \cup (A \cap B).$$

[10]

END OF EXAMINATION