

**UNIVERSITY OF SWAZILAND**

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**SUPPLEMENTARY EXAMINATIONS 2011/2012**

**B.Sc. / B.Ed. / B.A.S.S. II**

<u>TITLE OF PAPER</u>	:	FOUNDATIONS OF MATHEMATICS
<u>COURSE NUMBER</u>	:	M231
<u>TIME ALLOWED</u>	:	THREE (3) HOURS
<u>INSTRUCTIONS</u>	:	1. THIS PAPER CONSISTS OF <u>SEVEN</u> QUESTIONS. 2. ANSWER ANY <u>FIVE</u> QUESTIONS
<u>SPECIAL REQUIREMENTS</u>	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL  
PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) What do you understand by the following?

(i) Premiss; [1]

(ii) Deductive reasoning; [1]

(iii) Inductive reasoning; [1]

(iv) Mathematical Proof. [1]

(b) Using the axioms given below, prove each of the theorems which follow.

**Axiom 1** All mathematicians are logical.

**Axiom 2** Careful people are not foolish.

**Axiom 3** Discontented people are foolish.

**Axiom 4** Logical people are careful.

**Theorem 1** *Mathematicians are contented.*

**Theorem 2** *Foolish people are not logical.*

**Theorem 3** *Careless people are not mathematicians.*

[6]

(c) Consider the proposition "If  $z$  is a real number that satisfies  $z^3 + 3z^2 - 9z - 27 \geq 0$ , then  $|z| \geq 3$ ."

(i) Reword the proposition so that it is in the form " $A$  implies  $C$  OR  $D$ ." [2]

(ii) Prove the proposition by assuming that  $A$  and  $NOT C$  are true. [4]

(iii) Prove the proposition by assuming that  $A$  and  $NOT D$  are true. [4]

QUESTION 2

- (a) Write down the negation of the following statement: "The function  $f$  of one variable is a *convex function* if and only if for all real numbers  $x$  and  $y$  and for all real numbers  $t$  with  $0 \leq t \leq 1$ , it follows that  $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ ." [6]
- (b) Which of the following statements are true?
- (i) Some animals are four legged, whereas all the rest are two legged. [1]
  - (ii) The square root of any integer is a non-negative real number. [1]
- (c) For each of the following, write the *converse* and the *contrapositive*:
- (i) If  $n$  is an integer for which  $n^2$  is even, then  $n$  is even. [3]
  - (ii) Suppose that  $t$  is an angle between  $0$  and  $\pi$ . If  $t$  satisfies  $\sin(t) = \cos(t)$ , then  $t = \frac{\pi}{4}$ . [3]
- (d) Prove that the following statements are false:
- (i) For all  $n \in \mathbb{N}$ ,  $n^2 - n + 87$  is a prime number. [2]
  - (ii) For all  $n \in \mathbb{N}$ ,  $2n^2$  is an odd integer. [2]
  - (iii) For some  $n \in \mathbb{N}$ , with  $n \geq 2$ ,  $n^2 + 2n$  is a prime integer. [2]

### QUESTION 3

(a) Give the definition of a *valid argument*. [1]

(b) Using truth tables, analyze the following argument and then state whether it is valid or invalid.

“It is not true that he is rich and arrogant. He is rich. Therefore he is not arrogant.”

[6]

(c) Prove that if there are at least 6 people at a party, then either 3 of them knew each other before the party, or 3 of them were complete strangers before the party. [9]

(d) Show that the polynomial  $p(x) = x^4 - 2x^2 - 3$  has a root that lies between  $x = 1$  and  $x = 2$ . [4]

### QUESTION 4

(a) Let  $x$  be a rational number. Prove that there exists a monotone decreasing sequence  $(x_n)_{n \geq 0}$  of irrational numbers which converges to  $x$ . [10]

(b) Prove that there are infinitely many primes of the form  $3k + 2$ , where  $k$  is an integer. [10]

QUESTION 5

(a) State and prove the Principle of Strong Mathematical Induction. [6]

(b) The *Fibonacci sequence* is a sequence of integers  $u_1, u_2, \dots, u_n, u_{n+1}, \dots$ , such that  $u_1 = 1, u_2 = 1$  and

$$u_{n+1} = u_n + u_{n-1}$$

for all  $n \geq 2$ . The beginning of this sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Prove by strong induction, or by any other method, that for all positive integers  $n$ ,

$$u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

where  $\alpha = \frac{1 + \sqrt{5}}{2}$  and  $\beta = \frac{1 - \sqrt{5}}{2}$ .

[8]

(c) Suppose that Canada Post prints only 3 cent and 5 cent stamps. Prove that it is possible to make up any postage of  $n$  cents using only 3 cent and 5 cent stamps for  $n \geq 8$ . [6]

QUESTION 6

(a) Let  $A$  and  $B$  be sets. Prove that  $A \cup B = A$  if and only if  $B \subseteq A$ . [9]

(b) Prove that  $(A - C) \cap (B - C) = (A \cap B) - C$  for all sets  $A, B$  and  $C$ . [11]

QUESTION 7

- (a) Let  $X$  and  $Y$  be any sets, and let  $f : X \rightarrow Y$  be any mapping from  $X$  to  $Y$ .  
Let  $A$  and  $B$  be subsets of  $X$ . Prove that  $f(B) \setminus f(A) \subseteq f(B \setminus A)$  [4]
- (b) Let  $X$  and  $Y$  be any sets,  $A$  be a subset of  $X$ , and let  $f : X \rightarrow Y$  be an injection from  $X$  to  $Y$ . Prove that  $A = f^{-1}(f(A))$ . [4]
- (c) Let  $X$  and  $Y$  be any sets, and let  $f : X \rightarrow Y$  be a bijection from  $X$  to  $Y$ . Let  $A$  and  $B$  be subsets of  $Y$ . Prove that:
- (i)  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ ; [4]
- (ii)  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ ; [4]
- (iii)  $f^{-1}(B) \setminus f^{-1}(A) = f^{-1}(B \setminus A)$ . [4]

END OF EXAMINATION