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University of Swaziland

Final Examination, 2011/2012

BSc II, Bass II, BEd II

Title of Paper : Dynamics I

Course Number : M255

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

The position vector of a moving particle is given by

$$\vec{\mathbf{r}} = 4\sin t\ \hat{\mathbf{i}} + 4\cos t\ \hat{\mathbf{j}} + (3t-1)\hat{\mathbf{k}}.$$

Find

1.1 the velocity vector of the particle, (2)

1.2 the speed of the particle, (1)

1.3 the acceleration vector of the particle, (2)

1.4 the unit tangent vector $\hat{\mathbf{T}}$, (2)

1.5 the curvature κ of the path, (4)

1.6 the radius of curvature R of the path, (1)

1.7 the unit principal normal $\hat{\mathbf{N}}$, (2)

1.8 the normal component of acceleration, (3)

1.9 the binormal vector $\hat{\mathbf{B}}$. (3)

QUESTION 2

2.1 Let $\phi(x,y,z) = \sqrt{x^2+y^2+z^2}$. Show that $\operatorname{grad}\phi$ is the unit vector in the direction of $\vec{\mathbf{r}} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$. (4)

2.2 Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ have continuous second order partial derivatives. Prove that

$$\operatorname{curl}(\operatorname{grad}\phi) = \vec{\mathbf{0}}.$$

(5)

2.3 Let $\vec{\mathbf{F}}(x,y,z) = 2xy^2\ \hat{\mathbf{i}} + (2x^2y + 2yz^2)\ \hat{\mathbf{j}} + 2y^2z\ \hat{\mathbf{k}}$.

2.3.1 Verify that $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$. (4)

2.3.2 Find $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{\mathbf{F}} = \operatorname{grad}\phi$. (7)

QUESTION 3

3.1 Suppose a point A has position vector $\vec{\mathbf{a}}$ and a point B has position vector $\vec{\mathbf{b}}$. Show that the position vector $\vec{\mathbf{r}}$ of the point R, that divides the line AB in the ratio $\alpha : \beta$, is given by

$$\vec{\mathbf{r}} = \frac{\beta\vec{\mathbf{a}} + \alpha\vec{\mathbf{b}}}{\alpha+\beta}.$$

Hence deduce the midpoint formula. (6)

3.2 Show that $\vec{\mathbf{A}} = (6xy^2 - y^3)\,\hat{\mathbf{i}} + (6x^2y - 3xy^2)\,\hat{\mathbf{j}}$ is a conservative vector field. Hence, evaluate the integral

$$\int_{(1,2)}^{(3,4)} \vec{\mathbf{A}} \cdot d\vec{\mathbf{r}}.$$

(10

3.3 For what values of a are the vectors $\vec{\mathbf{A}} = a\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 2a\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ perpendicular? (4)

QUESTION 4

4.1 Two points A and B are at a distance d apart. A particle starts from A and moves with initial velocity u and uniform acceleration a in the direction $\overrightarrow{AB}$. A second particle starts out at the same time from B and moves with initial velocity $2u$ in the direction $\overrightarrow{BA}$ under uniform retardation a.

4.1.1 Show that the particles collide at a time $\dfrac{d}{3u}$ from the beginning of motion.

4.1.2 Show that for the particles to collide before the second particle returns to B, the inequality

$$ad < 12u^2$$

must hold.

(10

4.2 A car with initial speed u accelerates uniformly over a distance of $2s$ which it covers in time t_1. It is then stopped by being retarded uniformly to rest over a distance s, which it covers in time t_2. Show that

$$\frac{u}{2s} = \frac{2}{t_1} - \frac{1}{t_2}.$$

(10

QUESTION 5

5.1 In cylindrical coordinates (r, θ, z), the position vector of an arbitrary point (x, y, z) is given by

$$\vec{\mathbf{R}} = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}.$$

Show that in this coordinate system, the acceleration of a particle with position vector $\vec{\mathbf{R}}$ is given by

$$\vec{\mathbf{a}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{k}}.$$

(10

5.2 A particle of unit mass is thrown vertically upwards with initial speed V. The air resistance at speed v is kv^2 per unit mass where k is a constant. Show that the maximum height reached, H, is given by

$$H = \frac{1}{2k}\ln\left(\frac{g + kV^2}{g}\right).$$

(10

QUESTION 6

6.1 A particle is projected, under gravity, from a point O, which is at height h above sea level, with a velocity $\vec{\mathbf{v}}$ of magnitude $v = |\vec{\mathbf{v}}| = \frac{3}{2}\sqrt{gh}$. Find the two possible angles of projection if the particle strikes the sea at horizontal distance $3h$ from O. (10

6.2 Verify Green's theorem in the plane for the vector field $\vec{\mathbf{F}}(x,y) = (x^2 - xy^3)\,\hat{\mathbf{i}} + (y^2 - 2xy)\,\hat{\mathbf{j}}$ where the curve C is the boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$ oriented in the counterclockwise direction. (10

QUESTION 7

7.1 Show, by means of the substitution $r = 1/u$, that the equation of a particle in a central force field is

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2u^2}.$$

(10

7.2 Under the influence of a central force, a particle moves in a circular orbit through the origin. Find the law of force. (10

END OF EXAMINATION PAPER