UNIVERSITY OF SWAZILAND

84

FINAL EXAMINATIONS 2011/2012

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) (i) Find the point of intersection of the lines

$$x = 2t + 1, \qquad y = 3t + 2, \qquad z = 4t + 3$$

and

$$x = s + 2, \qquad y = 2s + 4, \qquad z = -4s - 1,$$

and then find the plane determined by these lines. [6]

(ii) Find the distance from the point $(0, -1, 0)$ to the plane $2x + y + 2z = 4$.[5]

(b) (i) The graph $y = f(x)$ in the xy-plane automatically has the parametrization $x = x$, $y = f(x)$, and the vector formula $\mathbf{r}(x) = x\hat{\mathbf{i}} + (f(x))\hat{\mathbf{j}}$. Use this formula to show that if f is a twice differentiable function of x, then

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

[7]

(ii) Use the formula for κ in (i) to find the curvature of $y = \ln(\cos x)$, $-\pi/2 \leq x \leq \pi/2$. [2]

QUESTION 2

(a) Show that $\mathbf{n}(t) = -g'(t)\hat{\mathbf{i}} + f'(t)\hat{\mathbf{j}}$ and $-\mathbf{n}(t) = g'(t)\hat{\mathbf{i}} - f'(t)\hat{\mathbf{j}}$ are both normals to the curve $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ at the point $(f(t), g(t))$. Hence find a unit normal, $\hat{\mathbf{N}}$, for the curve $\mathbf{r}(t) = \sqrt{4 - t^2}\,\hat{\mathbf{i}} + t\hat{\mathbf{j}}$, $\quad -2 \leq t \leq 2$. [6]

(b) Integrate $f(x, y, z) = 2x - 6y^2 + 2z$ over the line segment C joining the points (2,2,2) and (3,3,3). [6]

(c) Show that $y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z$ is exact and evaluate the integral

$$\int_{(2,2,2)}^{(3,4,0)} y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z.$$

[8]

QUESTION 3

(a) Give a formula $\mathbf{F} = M(x, y)\hat{\mathbf{i}} + N(x, y)\hat{\mathbf{j}}$ for the vector field in the plane with the properties that $\mathbf{F} = \mathbf{0}$ at the origin and that at any other point (a, b) in the plane, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction, with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$. [8]

(b) Verify the divergence theorem for $\mathbf{A} = (2x - z)\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - xz^2\hat{\mathbf{k}}$ taken over the region bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. [12]

QUESTION 4

(a) By any method, find the outward flux of the field $\mathbf{F} = (x + y)\hat{\mathbf{i}} + (x^2 + 2y + \sin z)\hat{\mathbf{j}} + (\mathrm{e}^x + \sqrt{y})\hat{\mathbf{k}}$ across the boundary of the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 3)^2 = 9$, and above by the paraboloid $z = x^2 + y^2$. [12]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2 + y^2)\hat{\mathbf{i}} + (x + y)\hat{\mathbf{j}}$ around the triangle with vertices (1,0), (0,1), (-2,0) traversed in the counterclockwise direction. [8]

QUESTION 5

(a) If $\mathbf{F} = y\hat{\mathbf{i}} + (x - 2xz)\hat{\mathbf{j}} - xy\hat{\mathbf{k}}$, evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. [12]

(b) Verify that the parametric equations

$$x = \rho^2 \cos\theta, \quad y = \rho^2 \sin\theta, \quad z = \rho$$

could be used to represent the surface $x^2 + y^2 - z^4 = 0$. Hence compute the unit normal to this surface at any point. [8]

QUESTION 6

(a) The path of a highway and an exit ramp are superimposed on a rectangular coordinate system such that the highway coincides with the x-axis. The exit ramp begins at the origin O. After following the graph of $y = -x^4/4$ from O to the point $P(1, -1/4)$, the path follows the arc of a circle in such a way that the ramp is *continuous*, *smooth*, and has *continuous curvature*. Find the equation of this circle. [12]

(b) Find the scale factors h_1, h_2, and h_3 in cylindrical and in spherical coordinates. Hence find the line element ds^2 and the volume element dV (in cylindrical and in spherical coordinates). [8]

QUESTION 7

(a) Find the arc length parameter along the curve $\mathbf{r}(t) = (e^t \cos t)\hat{\mathbf{i}} + (e^t \sin t)\hat{\mathbf{j}} + e^t\hat{\mathbf{k}}$, from the point where $t = 0$, by evaluating the integral

$$s = \int_{\tau=0}^{t} |\mathbf{v}(\tau)| \mathrm{d}\tau.$$

Then find the length of the portion of the curve in the closed interval $-\ln 4 \leq t \leq 0$. [6]

(b) Evaluate:

(i) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}} \mathrm{d}x$; and [3]

(ii) $\displaystyle\int_0^\infty \frac{y}{1+y^4} \mathrm{d}y$. [3]

(c) Use recurrence relations to show that

$$2J_0''(x) + J_0(x) - J_2(x) = 0,$$

where $J_n(x)$ is the Bessel function of the first kind of order n. [8]

END OF EXAMINATION