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UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2011/2012

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Show that

$$\beta(m,n) = 2\int_0^\infty \frac{\sinh^{2n-1}\theta}{\cosh^{2m+2n-1}\theta}\mathrm{d}\theta.$$

Hence, show that

$$\int_0^\infty \frac{\sinh^p\theta}{\cosh^q\theta}\mathrm{d}\theta = \frac{1}{2}\beta\Big(\frac{p+1}{2},\frac{q-p}{2}\Big).$$

[10]

(b) Legendre's differential equation is given by

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

Using this, or by any other method, prove that

$$\int_{-1}^1 P_m(x)P_n(x)\mathrm{d}x = 0, \quad \text{if } m \neq n.$$

[10]

QUESTION 2

(a) By any method, find the integral of $H(x,y,z) = yz$ over the part of the sphere $x^2+y^2+z^2 = 16$ that lies above the cone $z = \sqrt{x^2+y^2}$. [6]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\frac{x^2}{4}+\frac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x-4y)\hat{\mathbf{i}} + (4x+2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [4]

(c) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (yz^2)\hat{\mathbf{i}} + (xz^2)\hat{\mathbf{j}} + (x^2yz)\hat{\mathbf{k}}$. [2]

(ii) $\mathbf{F} = (e^x\sin y)\hat{\mathbf{i}} + (e^x\cos y + \sin z)\hat{\mathbf{j}} + (y\cos z)\hat{\mathbf{k}}$. [8]

QUESTION 3

(a) Verify Stokes' theorem for $\mathbf{A} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. [10]

(b) Evaluate $\iint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, where $\mathbf{A} = xy\hat{\mathbf{i}} - x^2\hat{\mathbf{j}} + (x+z)\hat{\mathbf{k}}$, S is that portion of the plane $2x + 2y + z = 6$ included in the first octant, and $\hat{\mathbf{n}}$ is the unit normal to S. [10]

QUESTION 4

(a) Find the plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersections of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. [7]

(b) Find the distance from the point $(3, -1, 4)$ to the line $x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$. [6]

(c) Find the point of intersection of the lines

$$x = 2t + 1, \qquad y = 3t + 2, \qquad z = 4t + 3$$

and

$$x = s + 2, \qquad y = 2s + 4, \qquad z = -4s - 1,$$

and then find the plane determined by these lines. [7]

QUESTION 5

(a) Let $\mathbf{u}(x,y,z) = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ and $\mathbf{v}(x,y,z) = \dfrac{\mathbf{u}}{(x^2+y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [1,1,1,2]

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [5,1]

(b) Determine the directional derivative of $\phi(x,y) = 100 - x^2 - y^2$ at the point (3,6) in the direction of the unit vector $\hat{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. [3]

(c) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 2$ at the point $P_0(1,1,3)$. [6]

QUESTION 6

(a) Verify the divergence theorem for $\mathbf{F} = (2x - z)\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - xz^2\hat{\mathbf{k}}\mathbf{i}$ taken over the region bounded by $x = 2$, $x = 5$, $y = 2$, $y = 5$, $z = 2$, $z = 5$. [10]

(b) Verify Green's theorem in the plane for

$$\oint_C [2x\mathrm{d}x - (3y - x)\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [10]

QUESTION 7

(a) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\mathbf{i}} + (2y + x^2z)\hat{\mathbf{j}} + (4x^2y^3)\hat{\mathbf{k}}$ across the boundary of the region cut from the first octant by the cylinder $x^2 + y^2 = 9$ and the plane $z = 9$. [10]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2+y^2)\hat{\mathbf{i}} + (x+y)\hat{\mathbf{j}}$ around the triangle with vertices (1,0), (0,1), (-3,0) traversed in the counterclockwise direction. [10]

END OF EXAMINATION