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University of Swaziland

Final Examination, 2011/12

BSc III, Bass III, BEd III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Show that if $(a, s) = 1$ and $(b, s) = 1$, then $(ab, s) = 1$ where $a, b, s \in \mathbb{Z}$. [6]

(b) Give a single numerical example to disprove the following:

"If $ax \equiv bx (\text{mod } n)$ then $a \equiv b (\text{mod } n)$ $\forall a, b, n \in \mathbb{Z}$"

[4]

(c) Prove that every subgroup of a cyclic group is cyclic. [10]

Question 2

(a) Solve the following system:

$$\begin{aligned} 2x &\equiv 1 (\text{mod } 5) \\ 3x &\equiv 4 (\text{mod } 7). \end{aligned}$$

[7]

(b) Find the number of generators for the cyclic groups of order 8 and 60. [7]

(c) Prove that a non-abelian group of order $2p$, p prime, contains at least one element of order p. [6]

Question 3

(a) Prove that every cyclic group is abelian. [5]

(b) Let n be a positive integer greater than 1 and let, for $a, b \in \mathbb{Z}$

$$aRb \Longleftrightarrow a \equiv b \ (\text{mod } n).$$

Prove that R is an equivalence relation on $\mathbb{Z}$. [7]

(c) Show that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$. [8]

Question 4

(a) Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 1 & 3 & 2 & 7 & 8 & 6 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 4 & 1 & 7 & 2 & 3 \end{pmatrix}.$$

(i) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one, [8]

(ii) Compute α^{-1}, $\beta^{-1}\alpha$, $(\alpha\beta)^{-1}$. [6]

(b) Find the greatest common divisor of the numbers 616 and 427 and express it in the form

$$d = 616x + 427$$

for some $x, y \in \mathbb{Z}$. [6]

Question 5

(a) Compute (do not list) the number of elements in each of the cyclic subgroups

(i) $< 30 >$ of $\mathbb{Z}_{42}$

(ii) $< 15 >$ of $\mathbb{Z}_{48}$

[6]

(b) For $\mathbb{Z}_{12}$, find all the subgroups and give a lattice diagram. [7]

(c)

(i) Find all cosets of $H = \{0, 6, 12\}$ in $\mathbb{Z}_{18}$.

(ii) Show that the groups $\mathbb{Z}_6$ and S_3 are not isomorphic. [7]

Question 6

(a) Prove that every finite group of prime order is cyclic. [5]

(b) Show that the set $G = \mathbb{Q} - \{0\}$ with respect to the operation

$$a * b = \frac{ab}{2}, \quad \forall\, a, b \in G$$

is a group. [9]

(c) Prove that if $\left(ab\right)^{-1} = a^{-1}b^{-1}\ \ \forall\, a, b \in G$, where G is a group, then G is abelian. [6]

Question 7

(a) If $\varphi : GH$ is an isomorphism of groups and e is the identity of G, then

(i) $(e)\varphi$ is the identity element in H

(ii) $(a^n)\varphi = \left[(a)\varphi\right]\ \ \forall\, n \in \mathbb{Z}^+$

[12]

(b)

(i) State Lagrange's theorem.

(ii) Using (b) (i) above, or otherwise, show that $\mathbb{Z}_p$ has no proper subgroup if p is a prime number.

[8]