UNIVERSITY OF SWAZILAND

102

FINAL EXAMINATION 2011/2012

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER : REAL ANALYSIS

COURSE NUMBER : M 331

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :
1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) If $0 < a < b$ then prove that $a^n < b^n$, $\forall n \in \mathbb{N}$. [5 marks]

   (b) Let S be a set of real numbers. Explain precisely each of the following statements.

      i. S is bounded above. [2 marks]

      ii. S is bounded below. [2 marks]

      iii. S is bounded. [2 marks]

   (c) Determine whether the set $S := \{x \in \mathbb{R} : |2x+1| > 5\}$ is bounded or not. [4 marks]

   (d) Let $\alpha > 0$ and let $T := \{\alpha s \in \mathbb{R} : s \in S\}$. Prove that $\sup(T) = \alpha \sup S$. [5 marks]

QUESTION 2

2. (a) Let (x_n) be a sequence of real numbers. Explain precisely each of the following statements.

      i. The sequence (x_n) is bounded. [2 marks]

      ii. The sequence (x_n) is monotone. [2 marks]

      iii. The sequence (x_n) is convergent. [2 marks]

   (b) Use your definition in 2(a)iii to prove that the sequence $\left(\dfrac{1+(-1)^n}{n}\right)$ converges to 0. [4 marks]

   (c) State the monotone convergence theorem for sequences of real numbers. [2 marks]

   (d) Consider the sequence (x_n) defined recursively by

$$x_1 = 2,\ 6x_{n+1} = x_n^2 + 5 \text{ for } n \geq 1$$

      i. Show that $1 < x_n < 5$, $\forall n \geq 1$. [3 marks]

      ii. Show that (x_n) is a decreasing sequence. [3 marks]

      iii. Deduce that (x_n) is convergent and find its limit. [2 marks]

QUESTION 3

3. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, and let $c \in (a, b)$.

   i. Explain precisely the statement "f is continuous at c". [2 marks]

   ii. Show that the constant function $f(x) \equiv d$ is continuous at c. [4 marks]

   iii. Prove that if both f and g are continuous at c then the difference $f - g$ is also continuous at c. [4 marks]

   iv. Give examples of functions f and g which make the converse of 3(a)iii false. [2 marks]

   (b) State the Intermediate value theorem and use it to show that the equation $15x^5 - 19x^3 - 1 = 0$ has a solution in the interval $[-1, 0]$. [5 marks]

   (c) Is the following statement true or false? Justify your answer.
   If a function $f : [0, 1] \to \mathbb{R}$ is continuous then so is the absolute value function $|f| : [0, 1] \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$ be a function.

   i. Explain the statement "f is differentiable at $c \in (a, b)$". [2 marks]

   ii. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} -x, & x \geq 0 \\ 1 - e^x, & x < 0 \end{cases}$$

   A. Show that f is differentiable at $x = 0$. [4 marks]

   B. Is f continuous at $x = 0$? Justify your answer. [2 marks]

   (b) i. State the Mean value theorem for derivatives. [2 marks]

   ii. Use the Mean value theorem for derivatives to prove each of the following statements.

   A. $|\sin x - \sin y| \leq |x - y|, \ \forall x, y \in \mathbb{R}$. [5 marks]

   B. The polynomial $p(x) = x^3 + ax + b$ (with $a > 0$) has exactly one real root. [5 marks]

QUESTION 5

5. (a) Let $\sum a_n$ be a series in $\mathbb{R}$. Precisely explain the following statements.

   i. $\sum a_n$ converges. [2 marks]

   ii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) Consider the series

$$\sum \frac{\sin n}{2n^2 - n} \tag{1}$$

   in $\mathbb{R}$

   i. Determine whether this series converges absolutely or not. State any theorems used. You may assume the result that the p-series $\sum \frac{1}{n^p}$ converges when $p > 1$. [4 marks]

   ii. Is the series in (1) convergent? Justify your answer. [2 marks]

   (c) Prove that if $\sum a_n$ converges then $\lim(a_n) = 0$. [4 marks]

   (d) Is the converse of 5c true? Justify your answer. [2 marks]

   (e) Let $\sum a_n$ be absolutely convergent, and let (b_n) be a bounded sequence of real numbers. Then, show that the series $\sum a_n b_n$ converges. [5 marks]

QUESTION 6

6. (a) Given that $f(x) := x, \forall x \in [2, 3]$, prove that the function f is integrable on $[2, 3]$ and find $\int_2^3 x$. [10 marks]

   (b) Show that if $f : [a, b] \to \mathbb{R}$ is a bounded, Riemann integrable function, then $F : [a, b] \to \mathbb{R}$ with $F(x) = \int_x^a f$ is a continuous function. [4 marks]

   (c) i. Let $D \subset \mathbb{R}$ be non-empty and let $f : D \to \mathbb{R}$ be a function. What does it mean to say that f is bounded on $[a, b]$? [2 marks]

   ii. State the boundedness theorem for integrals. [2 marks]

   iii. Is the converse of 2(a)iii true? Justify your answer. [2 marks]

QUESTION 7

7. (a) i. State the supremum property of $\mathbb{R}$. [2 marks]

ii. Let u be an upper bound for a non-empty subset V of $\mathbb{R}$. State a necessary and sufficient condition for u to equal $\sup V$. [2 marks]

iii. Let S and T be non-empty subsets of $\mathbb{R}$. Define
$S + T := \{x + y \in \mathbb{R} : x \in S, y \in T\}$.
Use your result of 7(a)ii above (or otherwise) to show that if both S and T are bounded above then $\sup(S + T) = \sup S + \sup T$. [6 marks]

(b) Determine whether each of the following statements is true or false. Justify your answer.

i. Every function $f : (0, 1) \to \mathbb{R}$ that is continuous on $(0, 1)$ is also bounded on $(0, 1)$. [2 marks]

ii. There are two distinct functions $f, g : [0, 1] \to \mathbb{R}$ such that the sum $f + g$ is Riemann integrable and yet neither f nor g is Riemann integrable. [2 marks]

iii. $\mathbb{N}$ is bounded above in $\mathbb{R}$. [2 marks]

iv. All divergent sequences are unbounded. [2 marks]

v. There is a function $f : [-1, 1] \to \mathbb{R}$ that is Riemann integrable on $[-1, 1]$ but not differentiable on $[-1, 1]$. [2 marks]