UNIVERSITY OF SWAZILAND

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FINAL EXAMINATION 2011/2012

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER : NUMERICAL ANALYSIS II

COURSE NUMBER : M 411

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :

1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS.
3. NON PROGRAMMABLE CALCULATORS MAY BE USED.

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let $w(x)$ be an integrable function and let $I = [a, b]$ be an interval in $\mathbb{R}$. Also, let $S := \{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ be a set of functions that are defined on I.

   i. What does it mean to say that w is a weight function on I? [2 marks]

   ii. What does it mean to say that S is orthogonal on I with respect to w? [2 marks]

   iii. What does it mean to say that S is linearly independent on I? [2 marks]

   (b) Suppose $w(x)$ is a weight function on closed interval $[a, b]$, and suppose

$$\int_a^b w = 1, \int_a^b wx = 2$$
$$\int_a^b wx^2 = 3, \int_a^b wx^3 = 6$$

   Take $\phi_0(x) = 1$.

   i. Determine polynomials $\phi_1(x)$ and $\phi_2(x)$ of degrees 1 and 2 respectively, so that $S := \{\phi_0, \phi_1, \phi_2\}$ is an orthogonal set on $[a, b]$ with respect to w. [10 marks]

   ii. Is S linearly independent on $[a, b]$? Justify your answer. [2 marks]

   (c) Give an example of a weight function $w(x)$ on the interval $(-1, 1)$. [2 marks]

QUESTION 2

2. (a) Find the linear least squares polynomial approximation to $f(x) = \ln(x+2)$ on $[1, 3]$. [10 marks]

   (b) Use Chebyshev polynomials of the first kind with degree at most 2 to approximate $\arccos(x)$. [10 marks]

QUESTION 3

3. (a) Approximate the integral $\int_0^{0.2} e^{-\tau^2} d\tau$ by using a single step of the Runge-Kutta method of order 2. [6 marks]

   (b) Use a single step of the Runge-Kutta method of order 2 to solve:

$$x'' + x' + 3x = te^t,\ 0 \leq x \leq 1,\ x(0) = 1, x'(0) = 0,$$

   for $x(0.1)$ and $x'(0.1)$ correct to 3 decimal places. [14 marks]

QUESTION 4

4. Consider the initial value problem (IVP) $x'(t) = f(t,x),\ a \le t \le b,\ x(a) = \alpha$

(a) Show that

$$x'(t_i) = \frac{-3x(t_i) + 4x(t_{i+1}) - x(t_{i+2})}{2h} + \frac{h^3}{3}x'''(\tau_i) \tag{1}$$

for some $\tau_i \in (t_i, t_{i+2})$. [4 marks]

(b) Let $h = 0.05$, $x_0 = x(t_0) = -1$, $x_1 = x(t_1) = 1 - e^{-0.05}$. Use the method

$$x_{i+2} = 4x_{i+1} - 3x_i - 2hf(t_i, x_i);\ i = 0, 1, \ldots, N-2$$

suggested by equation (1) to solve IVP; $x' = 1 + x,\ 0 \le t \le 1,\ x(0) = -1$ for $x(0.1)$. [3 marks]

(c) State the Dalquist equivalence theorem for the convergence of a multistep method. [2 marks]

(d) Analyse this method for consistency, zero-stability and convergence. [11 marks]

QUESTION 5

5. Let Ω be the $L-$shaped region in $\mathbb{R}^2$ enclosed by the polygonal path Γ passing through the points $(0,0), (0,1), (1,1), (1,3), (3,3)$ and $(3,0)$.

(a) Consider the differential problem

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0,\ (x,y) \in \Omega \\ u(x,y) &= x + y, (x,y) \in \Gamma \end{aligned}$$

Use the "*the* 5 *point formula*" with a uniform grid on Ω to approximate both $u(2,1)$ and $u(2,2)$. [10 marks]

(b) Given the Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = xy,\ (x,y) \in \Omega$$

subject to boundary condition

$$u(x,y) = x + y,\ (x,y) \in \Gamma,$$

consider a finite difference method resulting from using central difference approximations for the derivatives. Use this method to approximate both $u(2,1)$ and $u(2,2)$. [10 marks]

QUESTION 6

6. Consider the differential problem;

$$u_t(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0,$$
$$u_x(0,t) = 1,\ u_x(1,t) = 0, t > 0,$$
$$u(x,0) = x(1-x),\ 0 \le x \le 1.$$

Suppose that an approximate solution to this problem is determined by replacing u_t with a forward difference, and that both u_x and u_{xx} are replaced by central differences.

(a) Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{u}_{j+1} = B\mathbf{u}_j + \mathbf{v},\ \text{where } j = 0, 1, \ldots$$

Identify the square matrix B, and the vectors $\mathbf{u}_j$ and $\mathbf{v}$. [12 marks]

(b) Use this numerical scheme with $\Delta t = 0.1$ and $\Delta x = 0.5$ to approximate $u(0.5, 0.1)$. [8 marks]

QUESTION 7

7. (a) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2}$$

for approximating the differential equation

$$u_t = u_{xx} \quad (2)$$

is unconditionally stable. [10 marks]

(b) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

for approximating the differential equation

$$u_t + au_x = 0,\ (a > 0)$$

is convergent provided $a\dfrac{\Delta t}{\Delta x} \le 1$. [10 marks]