UNIVERSITY OF SWAZILAND

120

SUPPLEMENTARY EXAMINATION 2011/2012

BSc/ BEd/B.A.S.S IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. NON PROGRAMMABLE CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Show that the Chebyshev polynomials $\{T_0(x), T_1(x), \dots\}$ of the first kind are orthogonal on the open interval $(-1, 1)$ with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$. [10 marks]

   (b) Suppose $w(x)$ is a weight function on closed interval $[a, b]$, and suppose

$$\int_a^b w = 1, \int_a^b wx = 3$$
$$\int_a^b wx^2 = 4, \int_a^b wx^3 = 8$$

   Take $\phi_0(x) = 1$.

   Determine polynomials $\phi_1(x)$ and $\phi_2(x)$ of degrees 1 and 2 respectively, so that $S := \{\phi_0, \phi_1, \phi_2\}$ is an orthogonal set on $[a, b]$ with respect to w.

   [10 marks]

QUESTION 2

2. (a) Find the linear least squares polynomial approximation to $f(x) = xe^x$ on $[-1, 1]$. [8 marks]

   (b) Use Legendre polynomials of degree at most 2 to approximate e^x.

   [12 marks]

QUESTION 3

3. (a) Use a single step of the modified Euler method to solve:

$$x'' + 2x' + x = t \ln t,\ 0 \leq x \leq 1,\ x(0) = 0, x'(0) = 1,$$

   for $x(0.1)$ and $x'(0.1)$ correct to 3 decimal places. [14 marks]

   (b) Approximate the integral $\int_0^{0.1} e^{\tau^2} d\tau$ by using a single step of the Taylor series method of order 2. Give your answer correct to 3 decimal places.

   [6 marks]

QUESTION 4

4. The initial value problem (IVP)

$$x'(t) = f(t, x), \ a \leq t \leq b, \ x(a) = \alpha$$

may be solved using each of the following multistep methods with $n = 0, 1, \ldots, N-2$.

(a) $x_{n+1} = 5x_{n-1} - 4x_n + 2h[f(t_n, x_n) + 2f(t_{n-1}, x_{n-1})]$

(b) $x_{n+1} = -x_n + 2x_{n-1} + \frac{h}{2}[5f(t_n, x_n) + f(t_{n-1}, x_{n-1})]$

Analyse each method for consistency, zero-stability and convergence. [20 marks]

QUESTION 5

5. Let Ω be the $L-$shaped region in $\mathbb{R}^2$ enclosed by the polygonal path Γ passing through the points $(0,0), (0,3), (1,3), (1,2), (3,2)$ and $(3,0)$.

(a) Consider the Laplace equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \ (x,y) \in \Omega$$

subject to boundary condition

$$u(x,y) = xy, \ (x,y) \in \Gamma$$

Use the "*the* 5 *point formula*" with a uniform grid on Ω to approximate both $u(1,1)$ and $u(2,1)$. [10 marks]

(b) Given the Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = x + y, \ (x,y) \in \Omega$$

subject to boundary condition

$$u(x,y) = xy, \ (x,y) \in \Gamma,$$

consider a finite difference method resulting from using central difference approximations for the derivatives. Use this method to approximate both $u(1,1)$ and $u(2,1)$. [10 marks]

QUESTION 6

6. Consider the differential problem;

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) &= 1,\ u_x(1,t) = 0, t > 0, \\ u(x,0) &= \sin(\pi x),\ 0 \le x \le 1. \end{aligned}$$

Suppose that an approximate solution to this problem is determined by replacing u_t with a backward difference, and that both u_x and u_{xx} are replaced by central differences.

(a) Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{u}_j = B\mathbf{u}_{j-1} + \mathbf{v}, \text{ where } j = 1, 2, \ldots$$

Identify the square matrix B, and the vectors $\mathbf{u}_j$ and $\mathbf{v}$. [12 marks]

(b) Use this numerical scheme with $\Delta t = 0.1$ and $\Delta x = 0.5$ to approximate $u(0.5, 0.1)$. [8 marks]

QUESTION 7

7. (a) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

for approximating the differential equation

$$u_t = u_{xx} \tag{1}$$

is stable provided $0 < \frac{k}{h^2} \le \frac{1}{2}$. [10 marks]

(b) Determine the coefficients c_0, c_1 and c_{-1} so that the scheme

$$U_j^{n+1} = c_{-1}U_{j-1}^n + c_0U_j^n + c_1U_{j+1}^n$$

for approximating the differential equation

$$u_t + au_x = 0$$

agrees with the Taylor series expansion of $u(x_n, t_{n+1})$ to as high an order as possible when $a > 0$ is constant. [10 marks]