University of Swaziland

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Final Examination, December 2011

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions. **Submit solutions to ONLY FIVE questions.**
4. Show all your working.
5. A Table of Laplace Transforms is provided at the end of the question paper.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Find the general solution for the following pde

$$(x+y)(xu_x - yu_y) = (x^2 - y^2)(2xy - u).$$

[10 marks]

(b) Find a particular solution of the given pde that passes through the given curve Γ

$$3yu_x - 2xu_y = 0, \qquad x \neq 0,\ y \neq 0$$
$$\Gamma: \ u = 2x \ \text{ on } \ 3y^2 = 4$$

[10 marks]

Question 2

(a) By eliminating the arbitrary function, find the pde satisfied by

$$u(x,y) = e^{ax} f(bx + cy + u^3).$$

[6 marks]

(b) Determine the regions in which the pde

$$xu_{xx} + \sqrt{y}u_{xy} + xu_{yy} = e^x$$

is elliptic, hyperbolic and parabolic. [6 marks]

(c) Reduce the following pde to canonical form and find the general solution

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

[8 marks]

Question 3

(a) Find a relationship between a and b if $u = f(ax + by)$ is a solution of the pde

$$3u_x - 7u_y = 0$$

for any differentiable function f. [6 marks]

(b) Show that

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial F}{\partial \xi}\right)^2 + \frac{1}{\xi^2}\left(\frac{\partial F}{\partial \eta}\right)^2$$

under the transformation $x = \xi \cos \eta$, $y = \xi \sin \eta$. [14 marks]

Question 4

Consider the function

$$f(x) = \begin{cases} \pi + x, & -\pi \le x < 0; \\ \pi, & x = 0; \\ \pi - x, & 0 < x \le \pi. \end{cases} , \qquad f(x+2\pi) = f(x)$$

(a) Find the fourier series expansion for $f(x)$. [10 marks]

(b) Use Parseval's identity to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

[10 marks]

Question 5

Solve the following initial value value problem

$$\begin{aligned} &u_t = u_{xx}, \quad 0 < x < 30, \quad t > 0 \\ &u(0,t) = 20, \quad u(30,t) = 50, \quad t > 0 \\ &u(x,0) = 60 + x, \quad 0 < x < 30 \end{aligned}$$

using the method os separation of variables. [20 marks]

Question 6

Solve the following equations using the method of Laplace transforms

(a)

$$\begin{aligned} &u_{xt} + \sin t = 0, \quad x > 0, \quad t > 0 \\ &u(x,0) = x, \quad x \ge 0 \\ &u(0,t) = 0, \quad t \ge 0 \end{aligned}$$

[10 marks]

(b)

$$\begin{aligned} &x u_x + u_t = x, \quad x > 0, \quad t > 0 \\ &u(x,0) = 0, \quad x \ge 0 \\ &u(0,t) = 0, \quad t \ge 0 \end{aligned}$$

[10 marks]

Question 7

Solve the Dirichlet problem inside the circle

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \quad -\pi < \theta < \pi$$

$$u(1,\theta) = \begin{cases} 1, & -\pi < x < 0; \\ 2, & 0 < x < \pi. \end{cases}$$

[20 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$