University of Swaziland

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Supplementary Examination, July 2012

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations

Course Number : M415

Time Allowed : Three (3) Hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions. **Submit solutions to ONLY FIVE questions.**
4. Show all your working.
5. A Table of Laplace Transforms is provided at the end of the question paper.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

Consider the equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

(a) Find the characteristics. [3 marks]

(b) Reduce the pde to its canonical form. [8 marks]

(c) Hence or otherwise show that the solution is given by

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

[9 marks]

Question 2

(a) Determine the region in which the given pdes are elliptic, hyperbolic and parabolic

(i) $\sin^2 x u_{xx} - 2y \sin x u_{xy} + y^2 u_{yy} = 0.$ [3 marks]

(ii) $e^x u_{xx} + e^y u_{yy} + e^{x+y} u_x = 0.$ [3 marks]

(b) Find the characteristics for the pde

$$x u_{xx} + u_{yy} = x^2.$$

[6 marks]

(c) Reduce the following pde to canonical form and find the general solution

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0.$$

[8 marks]

Question 3

Consider the function

$$f(x) = \begin{cases} -1, & -\pi \le x < 0; \\ 0, & x = 0; \\ +1, & 0 < x \le \pi. \end{cases} , \qquad f(x+2\pi) = f(x)$$

(a) Find the fourier series expansion for $f(x)$. [10 marks]

(b) Use Parseval's identity to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

[10 marks]

Question 4

Solve the initial value problem

$$\begin{aligned} &u_{tt} = c^2 u_{xx} && 0 < x < 1, \quad t > 0 \\ &u(x,0) = 0, && 0 \le x \le 1 \\ &u_t(x,0) = 0, && 0 \le x \le 1 \\ &u(0,t) = u(1,t) = 0, && t \ge 0 \end{aligned}$$

using the method of separation of variables. [20 marks]

Question 5

(a) Solve the following pde using the method of Laplace transforms

$$\begin{aligned} &u_t = u_x + u, && x > 0, \quad t > 0 \\ &u(x,0) = e^{-4x}, && x \ge 0 \\ &\lim_{x \to \infty} |u(x,t)| < \infty, && t > 0 \end{aligned}$$

[12 marks]

(b) Find the general solution for the following pde

$$4u_x + 8u_y - u = 1.$$

[8 marks]

Question 6

Find the general solution for each of the following pdes

(a) $u(u_x - u_y) = u^2 + (x+y)^2$. [10 marks]

(b) $(xu_x - yu_y)u = y^2 - x^2$. [10 marks]

Question 7

Solve the following initial value problem

$$\begin{aligned}
&u_t - ku_{xx} = 0, \quad 0 < x < L, \ t > 0 \\
&u(0,t) = u(L,t) = 0, \ t \geq 0 \\
&u(x,0) = f(x), \ 0 \leq x \leq L
\end{aligned}$$

where L is a constant and $f(x)$ is an arbitrary function. [20 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$