UNIVERSITY OF SWAZILAND

137

FINAL EXAMINATIONS 2011/2012

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set with a map $d : X \times X \longrightarrow \mathbb{R}$. What is meant by saying that (X, d) is a *metric space*? [4]

(b) Check carefully that the set $X = \mathbb{C}$, equipped with the function d defined, for any $x, y \in \mathbb{C}$, by

$$d(x,y) = \begin{cases} \min\{|x| + |y|, |x-1| + |y-1|\} & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

is a metric on $\mathbb{C}$. [16]

QUESTION 2

(a) Give the definition of a pseudometric. [4]

(b) Let (X, d) be a metric space. Given any four points $x, y, z, t \in X$, prove that $|d(x,y) - d(z,t)| \leq |d(x,z) + d(y,t)|$. [6]

(c) A translation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a map given by $T(x) = (x_1 + a, x_2 + b)$ for some fixed point $(a, b) \in \mathbb{R}^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$. Prove that the Euclidian metric d_2 on $\mathbb{R}^2$ is translation invariant, in the sense that for any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we have

$$d_2(T(x), T(y)) = d_2(x, y).$$

[5]

(d) In each of the following cases, state with careful justification whether (X, d) is a metric space:

(i) $X = \mathbb{Q}$ equipped with $d(x,y) = (x-y)^3$;

(ii) $X = \mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ equipped with $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. [2,3]

QUESTION 3

(a) Let $X = \mathcal{C}[-1,1]$, and let $x(t) = t$ and $y(t) = t^3$ for $t \in [-1,1]$. Find $d(x,y)$ in $\mathcal{C}[-1,1]$, where d is the

(i) uniform metric,

(ii) L_1–metric, [3,2]

(b) Give an example of a subset A of $\mathbb{R}$ (equipped with the usual metric) such that $\text{diam}(A^\circ) < \text{diam}(A)$. [3]

(c) Let A be an open subset of a metric space (X,d), and let $a \in A$. Is the set $A \setminus \{a\}$ open or closed in X? Justify your answer. [3]

(d) Let Y be a subspace of the metric space X. Prove the following:

(i) $B \subseteq Y$ is open in Y if and only if $B = Y \cap A$ for some open set A in X;[6]

(ii) $B \subseteq Y$ is closed in Y if and only if $B = Y \cap F$ for some closed set in X.[3]

QUESTION 4

(a) Let (X,d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\frac{n^3}{2n^3+1}, \frac{1}{n+2}\sin(\frac{n\pi}{2})\right)$,

(ii) $x_n = (3^{-2n}, (-1)^n \exp(\frac{1}{n}))$. [4,4]

(c) (i) Suppose that (x_n) converges to x in $C\,[a,b]$ in the uniform metric. Explain what is meant by *pointwise convergence*. Show that (x_n) converges to x pointwise. [2,3]

(ii) Let x_n in $C[0,1]$ be defined by

$$x_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } 0 < t \leq 1. \end{cases}$$

Deduce that (x_n) is not convergent in $C[0,1]$ with the uniform metric. [1,2,2]

QUESTION 5

(a) Given a function $f : (X, d_1) \longrightarrow (X, d_2)$,

(i) When is f said to be continuous at a point $x_0 \in X$ in the $\varepsilon - \delta$ sense? [3]

(ii) Give an equivalent definition in terms of open sets. [4]

(iii) Assuming f is continuous at x_0, prove that

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

[6]

(b) Prove that the function $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\pi(x, y) = x$ is continuous when $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with their usual metrics. Is π uniformly continuous? Justify your answer. [7]

QUESTION 6

(a) Can you find metric spaces (X, d) where

(i) $[0, 1]$ is both closed and open? [2]

(ii) $[0, \frac{1}{2})$ is open but not closed? [2]

(b) (i) Let X be a metric space. Using the definition that a set is *open* if its complement is closed, prove that $A \subseteq X$ is open if and only if for every $a \in A$ there is an $r > 0$ such that the open ball $B(a, r) \subseteq A$. [4]

(ii) Let $X = \mathcal{C}[-1, 1]$. By considering the point $a(t) \equiv 1$ (i.e. $a(t) = 1 \; \forall \; t \in [-1, 1]$) in $\mathcal{C}[-1, 1]$, deduce that $A = \{x \in \mathcal{C}[-1, 1] : x(0) = 1\}$ is not open in $\mathcal{C}[-1, 1]$ with the uniform metric. [3]

(c) Let X be a nonempty set and let ρ and σ be metrics on X. We say that ρ and σ are *equivalent* if there exist positive constants α and β such that

$$\alpha \leq \frac{\rho(x, y)}{\sigma(x, y)} \leq \beta \text{ for all } , y \in X \text{ with } x \neq y.$$

Prove that if ρ and σ are equivalent metrics on X, then (X, ρ) and (X, σ) have the same open sets. [5]

(d) Let $X = (\mathbb{R}, d)$, and let $A = \bigcup_{n \in \mathbb{Z}_{\geq 0}} (n, n + 1)$, where $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. Sketch the set A, and decide whether A is an open subset, or a closed subset, or neither, of $\mathbb{R}$. Then find A°, $\overline{A}$, and $\partial(A)$. [4]

QUESTION 7

(a) Let X be a metric space. When is a subset $M \subseteq X$ said to be:

(i) bounded; [1]

(ii) totally bounded. [2]

(b) Define compactness of a metric space in terms of

(i) open coverings, [1]

(ii) sequences. [2]

(c) Show that a compact set is closed and bounded. [8]

(d) Which of the following sets is compact? Give reasons.

(i) $\{(x, y) : 0 \leq x \leq y \leq 1\}$ in $\mathbb{R}^2$, [3]

(ii) $\{1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^n}, \ldots\}$ in $\mathbb{R}$, where $n \in \mathbb{N}$. [3]

END OF EXAMINATION