UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2012/13

BSc./B.Ed./B.A.S.S II/B.ENG II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Use Crammer's rule to solve the following

$$\begin{aligned} 2x_1 + 8x_2 + x_3 &= 10 \\ 3x_2 - x_1 + 2x_3 &= -2 \\ 4x_1 + 4x_2 - 5x_3 &= 4 \end{aligned}$$

[5 marks]

(b) USe Gaussian elimination to solve the following

$$\begin{aligned} x - 2y + z &= 1 \\ x - 4y + 7z &= -15 \\ x - 4y - 11z &= 39 \end{aligned}$$

[5 marks]

(c) By inspection find the inverses of the following matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[4 marks]

(d) Verify Cayley-Hamilton theorem for the following matrix

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

[6 marks]

QUESTION 2

(a) Let $S = \{\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n\}$ be a set of non-zero vector space V. Prove that S is linearly dependent if and only if one of the vectors $\gamma_j \in s$ is a linear combination of the remaining vectors in S.

[10 marks]

(b) Evaluate the following determinant by expanding along the second row

$$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 2 \end{vmatrix}$$

[4 marks]

(c) Prove that if A and B are both invertible matrices then AB and BA are aslo invertible and $(AB)^{-1} = B^{-1}A^{-1}$ and $(BA)^{-1} = A^{-1}B^{-1}$ [6 marks]

QUESTION 3

(a) For which values of k does the following system has

(i) a unique solution

(ii) infinitely many solutions

(iii) no solution

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (k^2 - 5)z &= k \end{aligned}$$

[10 marks]

(b) Give the definition of a basis of a vector space V. [2 marks]

(c) Determine whether the vectors $\gamma_1 = (1,1,1)$ $\gamma_2 = (1,2,3)$ $\gamma_3 = (2,-1,1)$ form a basis for R^3

[8 marks]

QUESTION 4

(a) Prove that if a homogeneous system has more unknowns than the number of equations, than it always has a non-trivial solution.

[10 marks]

(b) Let $B = \{u_1, u_2, u_3\}$ and $B^1 = \{\gamma_1, \gamma_2, \gamma_3\}$

be bases in R^3 where

$u_1 = (0, 2, 1) \quad u_2 = (1, 0, 2) \quad u_3 = (1, -1, 0)$

and

$\gamma_1 = (1, 0, 0) \quad \gamma_2 = (1, 1, 0) \quad \gamma_3 = (1, 1, 1).$

Find the transition matrix from the basis B^1 to the basis B.

[10 marks]

QUESTION 5

(a) Show that the set $V = R^2$ with addition defined by

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$

and scalar multiplication defined by

$\alpha(x, y) = \alpha x_1 - \alpha - 1, \alpha y_1 - \alpha - 1)$ form a vector space.

[12 marks]

(b) Let $T : R^3 \rightarrow R^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y-z \end{pmatrix}$$

and let $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

and $B' = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

be bases for $\mathbb{R}^3$ and R^2 respectively. Find the matrix fo T with respect to B and B^1

[8 marks]

QUESTION 6

(a) Find the co-ordinate vector of $\begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$ with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \right\}$$

[6 marks]

(b) Find the characteristics polynomial eigenvalues and eigenvectors for the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{pmatrix}$$

[14 marks]

QUESTION 7

(a) Show that in the vector space $V = P_2(x)$, the vector $\gamma = x^2 + x + 2$ is a linear combination of the vectors

$$\gamma_1 = x^2 + 2x + 1 \quad \gamma_2 = x^2 + 3 \quad \gamma_3 = x - 1$$

[4 marks]

(b) Let $T : R^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \\ y-z \end{pmatrix}$$

(i) Find the matrix A of T with respect to the standard basis.

(ii) Find the matrix A^1 of T with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(iii) Find a 3×3 transition matrix P from teh standard basis to the basis B.

(iv) Give the relation between A adn A^1

[14 marks]