UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2012/2013

B.Sc. / B.Ed. / B.A.S.S. / B.EENG. III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Prove that the diagonals of a parallelogram bisect each other. [6]

(b) Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length. [6]

(c) Find the unit vectors that are tangent and normal to the curve $y = \int_0^x \sqrt{3+t^4}dt$ at the point $(0,0)$. [6]

(d) Prove that $\hat{\mathbf{i}}\cdot(\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = 1$. [2]

QUESTION 2

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space. Prove each of the following geometric properties of the cross product $\mathbf{u} \times \mathbf{v}$:

(i) $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane containing $\mathbf{u}$ and $\mathbf{v}$; [3]

(ii) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$; [3]

(iii) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are either parallel or antiparallel; and [2]

(iv) $|\mathbf{u} \times \mathbf{v}|$ is the are area of the parallelogram $ABCD$, where $\overrightarrow{\mathbf{AB}} = \mathbf{u}$ and $\overrightarrow{\mathbf{AD}} = \mathbf{v}$. [3]

(b) Let $\mathbf{F}(\mathbf{t}) = (\mathrm{e}^{\mathbf{t}}\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mathbf{t}^2\,\hat{\mathbf{k}}) \times (\mathbf{t}^3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$. Find $\mathbf{F}'(t)$. [4]

(c) Let $\mathbf{u}$ and $\mathbf{v}$ be space vectors. Prove *Lagrange's Identity*,

$$|\mathbf{u} \times \mathbf{v}|^2 = (|\mathbf{u}||\mathbf{v}|)^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

[5]

QUESTION 3

(a) Reparametrize the curve

$$x = \sin t, \quad y = \cos t, z = t; \qquad 0 \leq t \leq 2\pi,$$

in terms of the arc length. [4]

(b) Let C be the curve traced by $\mathbf{r} = \mathbf{r}(t)$; a suitably differentiable vector function. Show that:

(i) $\hat{\mathbf{B}} = \dfrac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}$; and [3]

(ii) $\hat{\mathbf{N}} = \dfrac{(\mathbf{r}'(t)\times\mathbf{r}''(t))\times\mathbf{r}'(t)}{|(\mathbf{r}'(t)\times\mathbf{r}''(t))\times\mathbf{r}'(t)|}$. [3]

(c) Find the outward unit normal vector to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b > 0,$$

at the point $P\left(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. [10]

QUESTION 4

(a) A curvilinear coordinate system (u, v, ϕ) is defined by

$$x = auv\cos\phi, \quad y = auv\sin\phi, \quad z = \frac{a}{2}(u^2 - v^2), \text{ where } u, v > 0, \ -\pi < \phi < \pi.$$

(i) Find the scale factors and the unit vectors. [6]

(ii) Show that the coordinate system is orthogonal. [3]

(iii) Find the line element and the volume element. [1,2]

(b) Find the maximum rate of increase of $f(x, y, z) = x + xyz$ at $(1, 3, -2)$. In what direction does this occur? [5]

(c) Find an equation for the tangent plane to the level surface $x^2 + yz = 5$ at $(2, 2, 1)$ [3]

QUESTION 5

(a) By any method, find the integral of $H(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$. [6]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x - 4y)\hat{\mathbf{i}} + (4x + 2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [4]

(c) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (yz^2)\hat{\mathbf{i}} + (xz^2)\hat{\mathbf{j}} + (x^2yz)\hat{\mathbf{k}}$. [2]

(ii) $\mathbf{F} = (e^x \sin y)\hat{\mathbf{i}} + (e^x \cos y + \sin z)\hat{\mathbf{j}} + (y \cos z)\hat{\mathbf{k}}$. [8]

QUESTION 6

(a) Evaluate $\iint_S [xz^2 \, dy \, dz + (x^2y - z^3) \, dz \, dx + (2xy + y^2z) \, dx \, dy]$, where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$

(i) by the divergence theorem (Green's theorem in space), [5]

(ii) directly. [7]

(b) Verify Stokes' theorem for $\mathbf{A} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. [8]

QUESTION 7

(a) Let $J_n(x)$ be the Bessel function of the first kind of order n. Evaluate

(i) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{\ln\frac{1}{x}}}$. [3]

(ii) $\displaystyle\int_0^\infty \sqrt{y}\,\exp^{-y^3}\,\mathrm{d}y$. [3]

(b) Using the Bessel function of the first kind ($J_n(x)$, express $J_4(ax)$ in terms of $J_0(ax)$ and $J_1(ax)$. [6]

(c) Legendre's differential equation is given by

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

Using this, or by any other method, prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0, \quad \text{if } m \neq n.$$

[8]

END OF EXAMINATION