UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2012/2013

B.Sc. / B.Ed. / B.A.S.S. / B.EENG. III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Show that the diagonals of a rhombus are perpendicular. [6]

(b) Show that squares are the only rectangles with perpendicular diagonals. [6]

(c) Find parametric equations for the tangent and normal lines to the curve C_1 : $x = t^2, \quad y = t^3, \quad z = 1 - t; \quad -\infty < t < \infty,$ at the point $(1, -1, 2)$. [5]

(d) Prove that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{u}) = 0$. [3]

QUESTION 2

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in space. Prove the *Parallelogram Law*,

$$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2).$$

[5]

(b) Prove that $\frac{\mathrm{d}}{du}(\mathbf{A}.\mathbf{B}) = \mathbf{A}.\frac{\mathrm{d}\mathbf{B}}{du} + \frac{\mathrm{d}\mathbf{A}}{du}.\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are differentiable functions of u. [5]

(c) Given the line

$$L: \quad x = 1 - 4t, \quad y = -5 - 2t, \quad z = -5 - 2t; \qquad -\infty < t < \infty,$$

find the equation of the plane containing L and the point $P_0(-3, 2, -2)$. [5]

(d) Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect. [5]

QUESTION 3

(a) Find the length of the arc between $(2,0,0)$ and $(0,2,0)$ of the curve curve traced by $\mathbf{r}(\theta) = 2\,\cos\theta\,\hat{\mathbf{i}} + 2\,\sin\theta\,\hat{\mathbf{j}};\ 0 \leq \theta \leq 2\pi$. [4]

(b) Let C be the curve traced by the vector function $a\,\cos t\,\hat{\mathbf{i}} + a\,\sin t\,\hat{\mathbf{j}} + bt\,\hat{\mathbf{k}}$, where a and b are positive numbers. Find the curvature of the curve C. [4]

(c) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (4,0) following a cubic curve. Find the equation of this curve if the track is *continuous*, *smooth*, and has *continuous curvature*. [12]

QUESTION 4

(a) Let $\mathbf{u}(x,y,z) = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ and $\mathbf{v}(x,y,z) = \dfrac{\mathbf{u}}{(x^2+y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [1,1,1,2]

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [5,1]

(b) Determine the directional derivative of $\phi(x,y) = 100 - x^2 - y^2$ at the point (3,6) in the direction of the unit vector $\hat{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. [3]

(c) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 2$ at the point $P_0(1,1,3)$. [6]

QUESTION 5

(a) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\mathbf{i}} + (2y + x^2z)\hat{\mathbf{j}} + (4x^2y^3)\hat{\mathbf{k}}$ across the boundary of the region cut from the first octant by the cylinder $x^2 + y^2 = 9$ and the plane $z = 9$. [10]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2+y^2)\hat{\mathbf{i}} + (x+y)\hat{\mathbf{j}}$ around the triangle with vertices (1,0), (0,1), (-3,0) traversed in the counterclockwise direction. [10]

QUESTION 6

(a) Verify the divergence theorem for $\mathbf{F} = (2x - z)\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - xz^2\hat{\mathbf{k}}\mathbf{i}$ taken over the region bounded by $x = 2$, $x = 5$, $y = 2$, $y = 5$, $z = 2$, $z = 5$. [10]

(b) Verify Green's theorem in the plane for

$$\oint_C [2x\mathrm{d}x - (3y - x)\mathrm{d}y],$$

where C is the closed curve (described in the positive direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [10]

QUESTION 7

(a) Evaluate

(i) $\int x^4 J_1(x)\mathrm{d}x$ [3]

(ii) $\int x^3 J_3(x)\mathrm{d}x$ [5]

(iii) $\int_0^\infty \frac{y^2}{1+y^4}\mathrm{d}y.$ [3]

(iv) $\int_0^\pi \sin^5\theta\mathrm{d}\theta,$ [4]

where $J_1(x)$ and $J_3(x)$ are Bessel functions of the first kind of order 1 and order 3, respectively.

(b) Evaluate

$$\int_0^\infty x^m \mathrm{e}^{-ax^n}\mathrm{d}x;$$

where m and n are positive integers. [5]

END OF EXAMINATION