UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2012/13

BSC III

TITLE OF PAPER	:	COMPLEX ANALYSIS
COURSE NUMBER	:	M313
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) (i) Prove that $Im(iz) = Re\ z$,

(ii) solve $z^n = 1$ for any integer n. [2,4]

(b) In the complex plane define and give example of

(i) domain,

(ii) interior point. [2,2]

(c) Sketch the following sets and determine which are domains

(i) $|2z + 3| > 4$,

(ii) $Re\ z \leq 2$,

(iii) $0 < arg\ z < \frac{\pi}{3}$. [2,2,2]

(d) Construct a line

$R_e \frac{1}{z} = 2$. [4]

QUESTION 2

(a) Find the region into which a transformation $w = z^2$ maps the strip $0 \leq x \leq c, \quad y \geq 0$. [4]

(b) Find the limits. Give your reasonings

(i) $\lim_{z \to \infty} \frac{3z + 3i}{1 + z}$,

(ii) $\lim_{z \to -2} \frac{iz + 2}{z + 2}$. [2,2]

(c) Define $f(z)$ continuous at z_o. [2]

(d) Using just a definition of derivative, find if possible $f'(z)$ if

(i) $f(z) = z^2 + 3$,

(ii) $f(z) = Im\ z$. [2,2]

(e) Derive Cauchy-Riemann conditions [6]

QUESTION 3

(a) Using Cauchy-Riemann equations (CRE)

(i) state the sufficient conditions theorem for existance of $f'(z)$, and thus

(ii) check if there is $f'(z)$ if $f(z) = z^2, \quad f(z) = |z|^2$. Find $f'(z)$. [2,6]

(b) Verify CRE for

$f(z) = z - \overline{z}$. [4]

(c) Use CRE in polars to show that

$f'(z) = e^{-i\theta}(u_r + iv_r)$

in the usual notations.

HINT: $u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$

$v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r}$. [8]

QUESTION 4

a) Consider $f(z) = \frac{1}{z}$. Give your reasonings to answer if f

(i) is analytic,

(ii) has the singular points. [2,2]

b) Prove that if $f(z) = u(x,y) + iv(x,y)$ is analytic, $z = x + iy$, in domain D, then u and v are harmonic in D. [6]

c) For the function $u(x,y) = x^2 + 3y^2$, find whether it can be the real part of a complex analytic function, and if so, find the corresponding imaginary part. [3]

d) Show that $u(x,y) = x^4 - 6x^2y^2 + y^4$ is harmonic and find corresponding harmonic conjugate function. [7]

QUESTION 5

a) Let $f(z) = z - 1$ and c is the arc from $z = 0$ to $z = 2$ consisting of the semicircle $z = 1 + e^{i\theta}$, $\pi \leq \theta \leq 2\pi$.

Evaluate $\int_c f(z)dz$. [6]

b) (i) Derive Cauchy formula for continuous $f'(z)$.

HINT: Apply Green's formula

$$\int_c Pdx + Qdy = \int\int_R (Q_x - P_y)dxdy.$$

(ii) Use result from (i) to evaluate

$$\int_c \frac{dz}{(z^2-1)(z^2+25)},$$

if $c = \{z : |z| = 4 \text{ in positive direction, and } |z| = 2 \text{ in negative direction}\}$. [6,3]

c) Apply Cauchy integral formula to evaluate

$$\int_c \frac{dz}{z^2+2z},$$

where c is a positively oriented circle $|z| = 1$. [5]

QUESTION 6

a) Apply extended Cauchy integral formula to evaluate

$\int_c \frac{dz}{(z^2+4)^2}$, where c is a positively oriented circle $|z - i| = 2$. [6]

b) State the Laurent series theorem. [3]

c) Expand $f(z) = \frac{1}{1+z}$ in Maclaurin series. [3]

d) i) Expand

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

in Laurent series in powers of $z - 1$ valid for $0 < |z - 1| < 2$.

(ii) What is the principal part of the series in i)? [7,1]

QUESTION 7

a) For $f(z) = \dfrac{1}{4z - z^2}$

(i) find residue at $z = 0$, and thus

(ii) evaluate $\displaystyle\int_c \frac{dz}{4z - z^2}$, where c is a positively oriented circle $|z| = 2$. [4,3]

b) i) State the residue theorem, and

ii) apply it to evaluate $\displaystyle\int_c \frac{dz}{1 + z^2}$, where c is a positively oriented circle $|z| = 2$. [2,4]

c) Using the residue theorem evaluate

$$\int_0^\infty \frac{x \sin x}{x^2 + 1} dx.$$