UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2012/13

BSC./B.ED./B.A.S.S III

TITLE OF PAPER	:	ABSTRACT ALGEBRA I
COURSE NUMBER	:	M323
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Suppose that d, a, b are positive integers, $(a, d) = 1$ and that d divides ab. Prove that d divides b. [5]

(b) Let H be a subgroup of a group G and let, for $a, b, \in G aRb$ if and only if $a = g^{-1}bg$ for some $g \in H$

Show that R is an equivalence relation on the set G. [6]

(c) The table below may be completed to define a binary opertion $*$ on the st $G = \{a, a, b, c\}$ in such a way that $(G, *)$ becomes a group. Assume this is possible and computer the missing entries.

$*$	e	a	b	c
e	e	a	b	c
a	a	e		b
b	b		a	
c	c	b		a

(d) Prove that, in any group G the indentity element is unique. [3]

QUESTION 2

(a) Determine whether the set $\mathbb{Q}$ with respect to the binary operation

$$a * b = a + b - 2013$$

is a group. [7]

(b) Find the greatest common divisor d of the numbers 102 and 42 and express it in the forms

$$d = 102m + 42n \quad \text{for some} \quad m, n \in \mathbb{Z}$$

[5]

(c) (i) State Lagrange's Theorem. [2]

(ii) Prove that every finite group prime order is cyclic. [6]

QUESTION 3

(a) Let H be the subset

$$\{\rho_0 = (1), \rho_1 = (1234), \rho_2 = (13)(24), \rho_3 = (1432)\}$$

of the group D_4.

(i) Show that H is a subgroup of D_4

(ii) Is H cyclic? Justify your answer. [10]

(b) Let $\phi : G \to H$ be an isomorphism of groups.

(i) Prove that, if e_g is the identity element of G, then $(e_g)\phi$ is the identity element of H.

(ii) Prove that, for any $a \in G$,

$$(a^{-1})\phi = [(a)\phi]^{-1}$$

[6]

(c) Determine all possible solutions

$$3x \equiv 5(mod \;\; 11)$$

[4]

QUESTION 4

(a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 1 & 7 & 5 & 3 & 4 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 8 & 3 & 4 & 5 & 2 & 6 \end{pmatrix}$

(a) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one. [7]

(b) Compute $\alpha^{-1}, \quad \beta^{-1}\alpha, \quad (\alpha\beta)^{-1}$ [7]

(c) Find the order of β and compute β^{2013}. [7]

QUESTION 5

(a) Prove that every subgroup of a cyclic group is cyclic. [8]

(b)Let H be the subgroup of $\mathbb{Z}_{20}$ generated by the element 8. i.e. $H = \langle 8 \rangle$. Find all cosets of H in $\mathbb{Z}_{20}$ [6]

(c) Prove that if G is a group and that $\forall \ a \in G, \quad a^2 = e$ then G is abelian. [6]

QUESTION 6

(a) Let G be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & o \\ b & c \end{pmatrix}$$

where $a, b, c, \in \mathbb{Q}, ac \neq 0$.

Show that, with respect to matrix multiplication, G is a group [8]

(b) Solve the system

$3x \equiv 2(mod\ \ 5)$

$2x \equiv 1(mod\ \ 3)$ [8]

(c) Give an example of a group satisfying the given conditions or, if there is no such group, say so (Do not prove anything)

(i) A finite non-abelian group

(ii) A non-abelian cyclic group.

[4]

QUESTION 7

(a) Find all subgroups of $\mathbb{Z}_{20}$ and draw a lattice diagram. [8]

b) (i) Define a subgroup of a group

(ii) Find the number of elements in the cyclic subgroup $\langle 30 \rangle$ of $\mathbb{Z}_{42}$ (Do not list the elements).

(c) Show that $\mathbb{R}$ under addition is isomorphic to $\mathbb{R}^+$ under multiplication. [12]