UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2012/2013

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Prove that $n < 3^n$, $\forall n \in \mathbb{N}$. [5 marks]

   (b) i. Find all $x \in \mathbb{R}$ that satisfy the inequality
   $4 < |x-2| + |x+1| < 5$. [4 marks]

   ii. Explain precisely the statement:
   "A non-empty set S of real numbers is bounded". [2 marks]

   iii. Let $S = \{x \in \mathbb{R} : 4 < |x-2| + |x+1| < 5\}$. Is S bounded? Justify your answer. [2 marks]

   (c) i. Let S be a non-empty subset of $\mathbb{R}$. Explain precisely each of the following statements.

   A. A real number u is an upper bound of S. [2 marks]

   B. A real number v is a supremum of S. [2 marks]

   ii. If $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S. [3 marks]

QUESTION 2

2. Let (x_n) be a sequence of real numbers.

   (a) i. Explain precisely the statement "(x_n) is convergent". [2 marks]

   ii. A. Prove that if (x_n) is convergent then $(|x_n|)$ is also convergent. [4 marks]

   B. Is the converse of 2(a)iiA true? Justify your answer. [2 marks]

   (b) i. Explain precisely each of the following statements.

   A. (x_n) is bounded. [2 marks]

   B. (x_n) is monotone. [2 marks]

   C. (x_n) is Cauchy. [2 marks]

   ii. State the monotone convergence theorem for (x_n). [2 marks]

   iii. Prove that if (x_n) is both bounded and monotone increasing then (x_n) is Cauchy. [4 marks]

QUESTION 3

3. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, and let $c \in (a, b)$.

    i. Explain precisely the statement "f is continuous at c". [2 marks]

    ii. Show that the absolute value function $f(x) := |x|$ is continuous at every point $c \in \mathbb{R}$. [4 marks]

    iii. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

    $$|f(x) - f(y)| \leq K|x - y|, \ \forall x, y \in \mathbb{R}$$

    Show that f is continuous at every point $c \in \mathbb{R}$. [4 marks]

    iv. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is **not** continuous at $x = \frac{1}{2}$. [2 marks]

   (b) State the Intermediate value theorem and use it to show that the equation $\cos x = x^2$ has a solution in the interval $[0, \pi/2]$. [5 marks]

   (c) Is the following statement true or false? Justify your answer.
   Given any 2 functions $f, g : [0, 1] \to \mathbb{R}$, if $f + g$ is continuous then so are both f and g. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$ be a function.

    i. Explain the statement "f is differentiable at $c \in (a, b)$". [2 marks]

    ii. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

    $$f(x) := \begin{cases} x + 1, & x \leq 0 \\ e^x, & x > 0 \end{cases}$$

    Show that f is differentiable at $x = 0$. [4 marks]

   (b) i. State the Mean value theorem for derivatives. [2 marks]

    ii. Use the Mean value theorem for derivatives to prove each of the following statements.

    A. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$, and that $f(0) = 0, f(1) = 1$.
    Show that $\exists c_1 \in (0, 1) : f'(c_1) = 1$. [2 marks]

    B. $-x \leq \sin x \leq x, \ \forall x > 0$. [5 marks]

    C. If $x > 1$ then $\dfrac{x - 1}{x} < \ln x < x - 1$. [5 marks]

QUESTION 5

5. (a) Let $\sum a_n$ be a series in $\mathbb{R}$. Precisely explain the following statements.

   i. $\sum a_n$ converges. [2 marks]

   ii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) Prove that if both $\sum x_n$ and $\sum y_n$ converge then $\sum (x_n + y_n)$ also converges. [4 marks]

   (c) Determine whether each of the following statements is true or false. Justify your answers.

   i. If $\sum a_n$ converges, then $\sum a_n$ converges absolutely. [2 marks]

   ii. If $\sum a_n$ with $a_n > 0$ converges, then $\sum \sqrt{a_n}$ converges. [2 marks]

   iii. If $\sum a_n$ converges, then $\sum a_n$ is absolutely convergent. [2 marks]

   (d) State the Cauchy convergence criterion for series. [2 marks]

   (e) Let (b_n) be a bounded sequence of real numbers. Show that if $\sum a_n$ is absolutely convergent, then the series $\sum a_n b_n$ converges. [5 marks]

QUESTION 6

6. (a) Determine whether each of the following statements is true or false. Justify your answer.

   i. If a function $f : [0,1] \to \mathbb{R}$ is bounded on $[0,1]$ then f is integrable on $[0,1]$. [2 marks]

   ii. If a function $f : [0,1] \to \mathbb{R}$ is integrable on $[0,1]$ then f is continuous on $[0,1]$. [2 marks]

   iii. If a function $f : [-1,1] \to \mathbb{R}$ is integrable on $[-1,1]$ then f is differentiable on $[-1,1]$. [2 marks]

   (b) Prove in detail that the function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$$

   is Riemann integrable and find $\int_0^2 f$. [10 marks]

   (c) Show that if $f : [a,b] \to \mathbb{R}$ is a bounded, Riemann integrable function, then $F : [a,b] \to \mathbb{R}$ with $F(x) = \int_x^a f$ is a continuous function. [4 marks]

QUESTION 7

7. (a) i. State the infimum property of $\mathbb{R}$. [2 marks]

ii. Let u be a lower bound for a non-empty subset V of $\mathbb{R}$. State a necessary and sufficient condition for u to equal $\inf V$. [2 marks]

iii. Let S and T be non-empty subsets of $\mathbb{R}$. Define
$S + T := \{x + y \in \mathbb{R} : x \in S, y \in T\}$.
Use your result of 7(a)ii above (or otherwise) to show that if both S and T are bounded below then $\inf(S + T) = \inf S + \inf T$. [6 marks]

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ and that $a, b \in \mathbb{R}$ with $a < b$. Let $g, h : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$\begin{aligned} g(x) &:= f(b) - f(x) - (b - x)f'(x), \\ h(x) &:= (b - a)^2 g(x) - (b - x)^2 g(a) \end{aligned}$$

i. Show that $h(a) = h(b)$. [2 marks]

ii. State Rolle's theorem. [2 marks]

iii. Use Rolle's theorem (or otherwise) to show that

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2}(b - a)^2 f''(c)$$

for some $c \in (a, b)$. [6 marks]