UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2012/2013

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let $a, b > 0$. Prove that $a < b \iff a^n < b^n, \forall n \in \mathbb{N}$. [5 marks]

   (b) i. Find all $x \in \mathbb{R}$ that satisfy the inequality $|x-2| \leq x+1$. [4 marks]

   ii. Explain precisely the statement:
   "A non-empty set S of real numbers is bounded". [2 marks]

   iii. Let $S = \{x \in \mathbb{R} : |x-2| \leq x+1\}$. Is S bounded?
   Justify your answer. [2 marks]

   (c) i. Let S be a non-empty subset of $\mathbb{R}$. Explain precisely each of the following statements.

   A. A real number u is an lower bound of S. [2 marks]

   B. A real number v is an infimum of S. [2 marks]

   ii. If $S \subseteq \mathbb{R}$ contains one of its lower bounds, show that this lower bound is the infimum of S. [3 marks]

QUESTION 2

2. (a) Let (x_n) be a sequence of real numbers. Explain precisely each of the following statements.

   i. (x_n) is convergent. [2 marks]

   ii. (x_n) is Cauchy. [2 marks]

   iii. (x_n) is bounded. [2 marks]

   (b) i. Let $(x_n), (y_n)$ be sequences of real numbers. Prove that if (x_n) and (y_n) are both Cauchy, then $(x_n + y_n)$ is also Cauchy. [4 marks]

   ii. Let $c \in \mathbb{R}$. Show that the constant sequence (c) is Cauchy. [4 marks]

   iii. Show directly from the definition that

   $$\left(\frac{n+1}{n}\right)$$

   is a Cauchy sequence. [4 marks]

   (c) Consider the statement:
   "Every bounded sequence of real numbers is convergent"
   Is this statement true or false? Justify your answer. [2 marks]

QUESTION 3

3. (a) Let $f, g : [a, b] \to \mathbb{R}$ be functions, and let $c \in (a, b)$.

   i. Explain precisely the statement "f is continuous at c". [2 marks]

   ii. Show that if both f and g are continuous at c, then the sum function $f + g$ is continuous at c. [4 marks]

   iii. Is the converse of (3(a)ii) true of false? Justify your answer. [2 marks]

   iv. Show that if both f and g are continuous at c, then the product function fg is continuous at c. [4 marks]

   v. Is the converse of (3(a)iv) true of false? Justify your answer. [2 marks]

   (b) State the Intermediate value theorem and use it to show that the equation $x^3 + 4x^2 - 5 = 0$ has a solution in the interval $[1, 2]$. [6 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$ be a function.

   i. Explain the statement "f is differentiable at $c \in (a, b)$". [2 marks]

   ii. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} e^x, & x \leq 0 \\ x + 1, & x > 0 \end{cases}$$

   A. Show that f is differentiable at $x = 0$. [4 marks]

   B. Is f continuous at $x = 0$? Justify your answer. [2 marks]

   (b) i. State the Mean value theorem for derivatives. [2 marks]

   ii. Use the Mean value theorem for derivatives to prove each of the following statements.

   A. $|\sin x - \sin y| \leq |x - y|$, $\forall x, y \in \mathbb{R}$. [5 marks]

   B. Let I be an interval. Let $f : I \to \mathbb{R}$ be differentiable on I. Show that if $f' > 0$ on I, then f is strictly increasing on I. [5 marks]

QUESTION 5

5. (a) Let $\sum a_n$ be a series in $\mathbb{R}$. Precisely explain the following statements.

   i. $\sum a_n$ converges. [2 marks]

   ii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) i. Prove that if $\sum x_n$ converges, then $\sum x_n$ is absolutely convergent. [4 marks]

   ii. Is the converse of 5(b)i true? Justify your answer. [2 marks]

   (c) If series $\sum x_n$ converges, then $\lim(x_n) = 0$. [4 marks]

   (d) State the Cauchy convergence criterion for series. [2 marks]

   (e) Let (a_n) be a bounded sequence of real numbers. Show that if $\sum a_n$ is absolutely convergent, then the series $\sum a_n^2$ converges. [5 marks]

QUESTION 6

6. (a) Let $f : [a, b] \to \mathbb{R}$. Use upper and lower sums to define the Riemann integral $\int_a^b f(x)dx$. [10 marks]

   (b) From the definition of the Riemann integral show that

$$\int_0^1 x^3\, dx = \frac{1}{4}$$

Assume without proof that

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{m(m+1)}{2}\right]^2, \forall n \in \mathbb{N}.$$

[10 marks]

QUESTION 7

7. (a) Use the Mean Value Theorem to show that;

   i. $\frac{1}{7} < \sqrt{38} - 6 < \frac{1}{6}$. [5 marks]

   ii. $\frac{1}{2} < \ln 2 < 1$. [5 marks]

   (b) i. State the supremum property of $\mathbb{R}$. [2 marks]

   ii. Let u be an upper bound for a non-empty subset V of $\mathbb{R}$. State a necessary and sufficient condition for u to equal $\sup V$. [2 marks]

   iii. Let S and T be non-empty subsets of $\mathbb{R}$. Define
   $S + T := \{x + y \in \mathbb{R} : x \in S, y \in T\}$.
   Use your result of 7(b)ii above (or otherwise) to show that if both S and T are bounded below then $\sup(S + T) = \sup S + \sup T$.[6 marks]