UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2012/2013

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. NON PROGRAMMABLE CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Given that $L_0(x) = 1$, use the Gram-schmidt process to construct an orthogonal set $\{L_0(x), L_1(x), L_2(x)\}$ of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$. [10 marks]

   (b) Use the so-called **Laguerre polynomials** of part 1a above to compute the least squares polynomial of degree two for approximating $f(x) = e^{-2x}$ on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$. [10 marks]

QUESTION 2

2. (a) Show that Chebyshev polynomials of the first kind sastisfy

$$T_i(x)T_j(x) = \frac{1}{2}[T_{i+j}(x) + T_{i-j}(x)]$$

   whenever i and j are any positive integers with $i > j$. [5 marks]

   (b) Show that if $S := \{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is an orthogonal set of functions on $[a, b]$ with respect to the weight function w, then S is a linearly independent set. [5 marks]

   (c) Find the linear least squares polynomial approximation to the function $f(x) = \frac{1}{2}\cos x + \frac{1}{3}\sin 2x$ on the interval $[0, 1]$. [10 marks]

QUESTION 3

3. (a) Show that Chebyshev polynomials of the second kind are orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$. [10 marks]

   (b) Table (1) contains data that shows the comparative crash-survivability characteristics of cars in various classes.

Type	Average Weight (Kg)	Percent occurrence
1. Domestic luxury regular	2400	3.1
2. Domestic intermediate regular	1850	4.0
3. Domestic ecomony regular	1700	5.2
4. Domestic compact	1400	6.4
5. Foreign compact	850	9.6

Table 1: % of accident-involved vehicles in which most severe injury was fatal

   Find the least squares line that approximates these data. [10 marks]

QUESTION 4

4. (a) State the Dalquist equivalence theorem for the convergence of a multistep method and use it to show that the differentiation formula

$$x_{k+2} - \frac{4}{3}x_{k+1} + \frac{1}{3}x_k = \frac{2}{3}hf(t_{k+2}, x_{k+2}) \qquad (1)$$

for determining a numerical solution to initial value problem (IVP)

$$x'(t) = f(t, x), \ a \leq t \leq b, \ x(a) = \alpha$$

is convergent. [14 marks]

(b) Solve the IVP

$$x' = 1 + x, \ 0 \leq t \leq 1, \ x(0) = -1$$

for $x(0.02)$ using formula (1) with $h = 0.01$ and starting values $x_0 = -1$ and $x_1 = 1 - e^{-0.01}$. [6 marks]

QUESTION 5

5. Consider the IVP

$$x'' + 2x' - 3x = 2t, \ x(0) = 1, x'(0) = 2, \qquad (2)$$

(a) Approximate both $x(0.1)$ and $x'(0.1)$ using one step of the 2-th order Runge-Kutta method. [14 marks]

(b) Compare your results against the exact values. [6 marks]

QUESTION 6

6. Let Ω be the $L-$shaped region in $\mathbb{R}^2$ enclosed by the polygonal path Γ passing through the points $(0,0),(1,0),(1,1),(3,1),(3,3)$ and $(0,3)$.

   (a) Solve the differential problem

$$\begin{aligned} u_{xx}(x,y)+u_{yy}(x,y) =&0,\ (x,y)\in\Omega \\ u(x,y) =&xy, (x,y)\in\Gamma \end{aligned}$$

   using the "*the* 5 *point formula*" with a uniform grid on Ω to approximate both $u(1,2)$ and $u(2,2)$. [10 marks]

   (b) Given the Poisson equation

$$u_{xx}(x,y)+u_{yy}(x,y) =x^2y,\ (x,y)\in\Omega$$

   subject to boundary condition

$$u(x,y) =(x+y)^2,\ (x,y)\in\Gamma,$$

   consider a finite difference method resulting from using central difference approximations for the derivatives. Use this method to approximate both $u(1,2)$ and $u(2,2)$. [10 marks]

QUESTION 7

7. (a) Under what condition in terms of h and k, is the numerical scheme

$$\frac{U_j^{n+1}-U_j^n}{k} =\frac{U_{j-1}^n-2U_j^n+U_{j+1}^n}{h^2} \tag{3}$$

   for approximating the differential equation

$$u_t =u_{xx} \tag{4}$$

   is stable? Prove your result. [10 marks]

   (b) Suppose $u(x,t)$ satisfies (4) whenever $0<x<1$ and $t>0$ subject to

$$\begin{aligned} u(0,t) =&u(1,t)=0,\ t>0, \\ u(x,0) =&x(1-x),\ 0\le x\le 1 \end{aligned}$$

   Use the numerical scheme (3) with $h=0.25$ and $k=0.01$ to approximate $u(0.25,0.01), u(0.5,0.01)$ and $u(0.75,0.01)$. [10 marks]