UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2012/2013

BSc/ BEd/B.A.S.S IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS.
		3. NON PROGRAMMABLE CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Find a function of the form $y = a + bx$ that best fits the data

i	0	1	2	3
x_i	1	2	3	4
y_i	4.1	5.3	7.2	6.9

in the least squares sense. [10 marks]

(b) Show that the Chebyshev polynomials of the first kind are orthogonal on the open interval $(-1, 1)$ with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$. [10 marks]

QUESTION 2

2. (a) Find the linear least squares polynomial approximation to $f(x) = \ln x$ on $[0, 1]$. [8 marks]

(b) Use Legendre polynomials of degree at most 2 to approximate e^{2x}.

[12 marks]

QUESTION 3

3. (a) Use a single step of the Taylor series method of order 2 to solve

$$x'' + 2x' + x = t \ln t,\ 0 \leq x \leq 1,\ x(0) = 0, x'(0) = 1,$$

for $x(0.1)$ and $x'(0.1)$ correct to 3 decimal places. [14 marks]

(b) Approximate the integral $\int_0^{0.1} e^{-\tau^2} d\tau$ by using a single step of the modified Euler method. Give your answer correct to 3 decimal places.

[6 marks]

QUESTION 4

4. The initial value problem (IVP)

$$x'(t) = f(t,x),\ a \le t \le b,\ x(a) = \alpha$$

may be solved using each of the following multistep methods.

(a) $y_{n+1} = y_n + \frac{h}{2}[f_{n-1} + f_n]$

(b) $y_{n+2} = -3y_n + 4y_{n+1} - 2hf_n$

Analyse each method for consistency, zero-stability and convergence.[20 marks]

QUESTION 5

5. Let Ω be the $L-$shaped region in $\mathbb{R}^2$ enclosed by the polygonal path Γ passing through the points $(0,0),(0,3),(1,3),(1,2),(3,2)$ and $(3,0)$.

(a) Consider the Laplace equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0,\ (x,y) \in \Omega$$

subject to boundary condition

$$u(x,y) = x^2 + y,\ (x,y) \in \Gamma$$

Use the "*the* 5 *point formula*" with a uniform grid on Ω to approximate both $u(1,1)$ and $u(2,1)$. [10 marks]

(b) Given the Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = xy,\ (x,y) \in \Omega$$

subject to boundary condition

$$u(x,y) = x + y,\ (x,y) \in \Gamma,$$

consider a finite difference method resulting from using central difference approximations for the derivatives. Use this method to approximate both $u(1,1)$ and $u(2,1)$. [10 marks]

QUESTION 6

6. Consider the differential problem;

$$\begin{aligned} u_t(x,t) =& u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) =& 1,\ u_x(1,t) = 0, t > 0, \\ u(x,0) =& \sin(\pi x),\ 0 \leq x \leq 1. \end{aligned}$$

Suppose that an approximate solution to this problem is determined by replacing u_t with a forward difference, and that both u_x and u_{xx} are replaced by central differences.

(a) Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{u}_{j+1} = B\mathbf{u}_j + \mathbf{v}, \text{ where } j = 0, 1, \ldots$$

Identify the square matrix B, and the vectors $\mathbf{u}_j$ and $\mathbf{v}$. [12 marks]

(b) Use this numerical scheme with $\Delta t = 0.1$ and $\Delta x = 0.5$ to approximate $u(0.5, 0.1)$. [8 marks]

QUESTION 7

7. (a) Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2}$$

for approximating the differential equation

$$u_t = u_{xx} \tag{1}$$

is unconditionally stable. [10 marks]

(b) Determine the coefficients c_0, c_1 and c_{-1} so that the scheme

$$U_j^{n+1} = c_{-1}U_{j-1}^n + c_0U_j^n + c_1U_{j+1}^n$$

for approximating the differential equation

$$u_t + au_x = 0$$

agrees with the Taylor series expansion of $u(x_n, t_{n+1})$ to as high an order as possible when $a > 0$ is constant. [10 marks]